Anwendung: lin. DGl.

~~Ein~~ System linearer DGl.en
1. Ordnung (homogen)
mit konstanten Koeffizienten

$$\begin{cases} \frac{dx_1(t)}{dt} = a_{11} x_1(t) + \ldots + a_{1n} x_n(t) \\ \vdots \\ \frac{dx_n(t)}{dt} = a_{n1} x_1(t) + \ldots + a_{nn} x_n(t) \end{cases}$$

$$x(t) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x : \mathbb{R} \to \mathbb{R}^n$$

$$\frac{dx}{dt} = Ax(t) \quad \forall t \in \mathbb{R}$$

gesucht: $x : \mathbb{R} \to \mathbb{R}^n$

gegeben: $A \in M(n, \mathbb{R})$,

Anfangsbed. $x(0) = a \in \mathbb{R}^n$.

$$\dot{x} = \frac{dx}{dt} \qquad x' \quad \dot{x}$$

Falls A diag.bar, $S^{-1}AS = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = D$,

dann gilt für $y(t) = S^{-1}x(t)$:

$$\underline{\underline{\dot{y}(t)}} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}$$

$$= \lim_{h \to 0} \frac{S^{-1} \cancel{g} x(t+h) - \cancel{s} S^{-1} \cancel{y} x(t)}{h}$$

$$= \lim_{h \to 0} S^{-1} \frac{x(t+h) - x(t)}{h}$$

$$= S^{-1} \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

Analysis 2 „jede lineare Abb. $\mathbb{R}^m \to \mathbb{R}^n$ ist stetig"

$$= S^{-1} \dot{x}(t)$$

$$= S^{-1} A x(t)$$

$$= S^{-1} A S y(t)$$

$$= \underline{\underline{D y(t)}} \qquad (\dot{y} = Dy)$$

$$\Longleftrightarrow \begin{cases} \dot{y}_1 = \lambda_1 y_1 \\ \vdots \\ \dot{y}_n = \lambda_n y_n \end{cases} \qquad \text{entkoppelt}$$

Anfangsdaten: $y(0) = S^{-1} a =: b$

$$\begin{cases} y_1(t) = b_1 e^{\lambda_1 t} \\ \vdots \\ y_n(t) = b_n e^{\lambda_n t} \end{cases}$$

$y_k(t) := b_k e^{\lambda_k t}$

$$\Rightarrow \frac{dy_k}{dt} = b_k \lambda_k e^{\lambda_k t} = \lambda_k y_k(t).$$

Anf. Bed. $y_k(0) = b_k e^0 = b_k$.

$$y(t) = e^{Dt} b := \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} b$$

Daher $x(t) = S y(t) = \underline{S e^{Dt} S^{-1}} a$

$$= e^{At} a$$

<u>Def</u> $$e^B := \sum_{k=0}^{\infty} \frac{B^k}{k!}$$

Wenn $A = SDS^{-1}$, dann

$$A^2 = SD\cancel{S^{-1}S}DS^{-1} = SD^2S^{-1}$$

$$A^3 = SD\cancel{S^{-1}S}D^2S^{-1} = SD^3S^{-1}$$

$$A^k = SD^kS^{-1}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} SD^kS^{-1}$$

$$= S\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k\right) S^{-1}$$

$$= S \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} \right) S^{-1}$$

$$= S \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} S^{-1} .$$

$$\frac{d}{dt} \underbrace{e^{At} a}_{x(t)} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k a$$

$$= \sum_{k=\cancel{0}\,1}^{\infty} \frac{\cancel{k}\, t^{k-1}}{(k-1)!\,\cancel{k}} A^k a$$

$$\overset{m=k-1}{=} \sum_{m=0}^{\infty} \frac{t^m}{m!} A^{m+1} a$$

$$= A \underbrace{\sum_{m=0}^{\infty} \frac{t^m}{m!} A^m}_{e^{At}} a$$

$$= A\, x(t) .$$