Kap. 6: Skalarprodukt

6.1 Def Sei V VR über $\mathbb{K} = \mathbb{R}$ oder $\mathbb{C}$

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{K}$$

heißt <u>Skalarprodukt</u> oder <u>inneres Produkt</u>, falls

a) $\langle u, u \rangle \geq 0 \quad \forall u \in V$ und

$$\langle u, u \rangle = 0 \iff u = 0$$

("positiv definit")

b) linear im 2ten Argument:

$$\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$$

$$\forall u, v, w \in V \quad \forall \alpha, \beta \in \mathbb{K}$$

c) $\langle u, v \rangle = \overline{\langle v, u \rangle}$

("symmetrisch / Hermitesch")

$(V, \langle \cdot , \cdot \rangle)$ heißt <u>Skalarproduktraum</u> oder Prä-Hilbertraum oder (für $\mathbb{K} = \mathbb{R}$) euklidischer VR.

6.2 Bem

- Für $\mathbb{K}=\mathbb{R}$ $\quad$ c) $\Leftrightarrow \langle u,v\rangle = \langle v,u\rangle$

$\overset{b)}{\Longrightarrow}$ linear auch im 1sten Arg.

also bilinear.

- Für $\mathbb{K}=\mathbb{C}$: antilinear

= semilinear

= konjugiert-linear

im 1sten Arg.

$$\langle \alpha u+\beta v, w\rangle \overset{c)}{=} \overline{\langle w, \alpha u+\beta v\rangle}$$

$$\overset{b)}{=} \overline{\alpha\langle w,u\rangle + \beta\langle w,v\rangle}$$

$$= \bar{\alpha}\, \overline{\langle w,u\rangle} + \bar{\beta}\, \overline{\langle w,v\rangle}$$

$$\overset{c)}{=} \bar{\alpha}\langle u,w\rangle + \bar{\beta}\langle v,w\rangle$$

D.h. $\langle\cdot,\cdot\rangle$ ist sesquilinear.

$V\times V \to \mathbb{K}=\mathbb{C}$ Sesquilinearform.

6.3 Bspe a) Standard-Skalarprod.

("Punktprodukt") in $\mathbb{R}^n$:

$$x\cdot y = \sum_{i=1}^{n} x_i\, y_i = \underset{1\times n}{x^T}\ \underset{n\times 1}{y}$$

f) Standard-Skalarprod. $\mathbb{C}^n$:

$$\langle x,y\rangle := \sum_{i=1}^{n} \overline{x_i}\, y_i = \underset{1\times n}{\overline{x}^T}\ \underset{n\times 1}{y}$$

Geometrische Bedeutung im $\mathbb{R}^n$

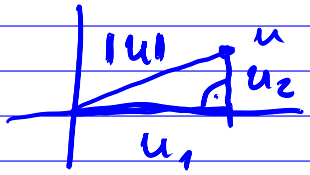
(i) "Norm von u"

$$\|u\| := \sqrt{\langle u,u\rangle} = \sqrt{\sum_{i=1}^{n} u_i^2}$$

= Länge des Vektors u

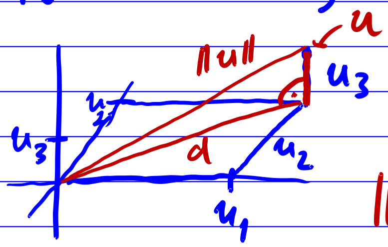
= Entfernung von u zu 0.

In $\mathbb{R}^2$: Satz von Pythagoras



$$\|u\|^2 = u_1^2 + u_2^2$$

In $\mathbb{R}^3$: 2x Satz von Pythagoras

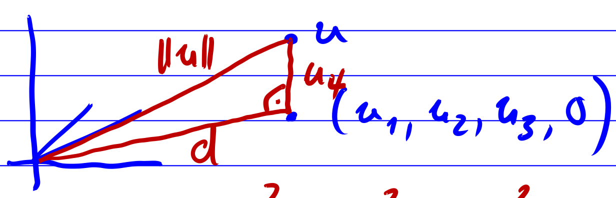


$$d = \sqrt{u_1^2 + u_2^2}$$

$$\|u\|^2 = d^2 + u_3^2$$

$$= u_1^2 + u_2^2 + u_3^2.$$

In $\mathbb{R}^4$



$$\|u\|^2 = d^2 + u_4^2$$

$$= u_1^2 + u_2^2 + u_3^2 + u_4^2.$$

etc.

Bem • $\|\alpha u\| = |\alpha|\,\|u\|$

"homogen vom Grad 1"

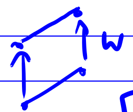
$$\|\alpha u\| = \sqrt{\langle \alpha u, \alpha u\rangle} = \sqrt{\bar\alpha \alpha \langle u, u\rangle} = \sqrt{\bar\alpha\alpha}\,\sqrt{\langle u, u\rangle} = |\alpha|\,\|u\|$$

$\left[\bar z z = (x-iy)(x+iy) = x^2 - ixy + ixy + y^2 = x^2+y^2\right]$

• $$\|u\|^{-1} u = \frac{u}{\|u\|} = \underbrace{\text{Einheitsvektor}}_{v \text{ mit } \|v\|=1} \text{ in Richtung } u \neq 0.$$

• $\text{Abstand}(u,v) = \|u-v\|$

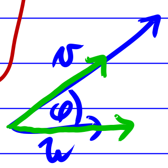
denn $\text{Abstand}(u+w, v+w) = \text{Abstand}(u,v)$



$[w := -v] \Rightarrow \text{Abst.}(u-v, 0)$

(ii) In $\mathbb{R}^n$:

$$\boxed{\langle u, v\rangle = \|u\|\cdot\|v\| \cos\varphi}$$



$$\Longleftrightarrow \left\langle \frac{u}{\|u\|}, \frac{v}{\|v\|}\right\rangle = \cos\varphi$$