Bem • $\|\alpha u\| = |\alpha|\,\|u\|$

"homogen vom Grad 1"

$$\|\alpha u\| = \sqrt{\langle \alpha u, \alpha u\rangle} = \sqrt{\bar\alpha \alpha \langle u, u\rangle} = \sqrt{\bar\alpha\alpha}\,\sqrt{\langle u,u\rangle} = |\alpha|\,\|u\|$$

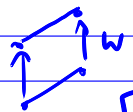
$\left[\bar z z = (x-iy)(x+iy) \quad x^2 - ixy + ixy + y^2 = x^2+y^2\right]$

• $\|u\|^{-1} u = \dfrac{u}{\|u\|}$

$= \underbrace{\text{Einheitsvektor}}_{v \text{ mit } \|v\|=1}$ in Richtung $u \neq 0$.

• $\text{Abstand}(u,v) = \|u-v\|$

denn $\text{Abstand}(u+w, v+w)$

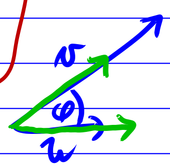


$= \text{Abstand}(u,v)$

$[w := -v] \Rightarrow \text{Abst.}(u-v, 0)$

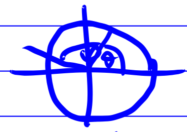
(ii) In $\mathbb{R}^n$:

$$\boxed{\langle u, v\rangle = \|u\|\cdot\|v\| \cos\varphi}$$



$$\Leftrightarrow \left\langle \frac{u}{\|u\|}, \frac{v}{\|v\|}\right\rangle = \cos\varphi$$

Geometrischer Beweis in $\mathbb{R}^2$:

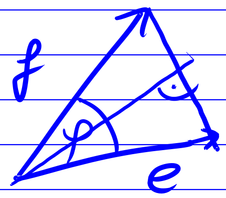


$$e := \frac{u}{\|u\|} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad f := \frac{v}{\|v\|} = \begin{pmatrix} \cos(\theta+\varphi) \\ \sin(\theta+\varphi) \end{pmatrix}$$

$$\begin{aligned}
\langle e, f\rangle &= \cos\theta \cos(\theta+\varphi) + \sin\theta \sin(\theta+\varphi) \\
&= \cos\theta \left(\cos\theta \cos\varphi - \cancel{\sin\theta \sin\varphi}\right) \\
&\quad + \sin\theta\left(\sin\theta\cos\varphi + \cancel{\cos\theta \sin\varphi}\right) \\
&= \cos^2\theta \cos\varphi + \sin^2\theta\cos\varphi \\
&= \cos\varphi .
\end{aligned}$$

Geometrischer Beweis in $\mathbb{R}^n$:

$$2\sin\frac{\varphi}{2} = \frac{\|e-f\|}{\cancel{1}}$$



$$\begin{aligned}
\Rightarrow \quad \underline{4\sin^2\frac{\varphi}{2}} &= \|e-f\|^2 \\
&= \langle e-f, e-f\rangle \\
&= \langle e, e-f\rangle - \langle f, e-f\rangle \\
&= \langle e,e\rangle - \langle e,f\rangle \\
&\quad - \langle f,e\rangle + \langle f,f\rangle \\
&= \underbrace{\|e\|^2}_{1} - 2\langle e,f\rangle + \underbrace{\|f\|^2}_{1} \\
&= \underline{2 - 2\langle e,f\rangle}
\end{aligned}$$

$$\langle e, f\rangle = 1 - 2\sin^2\frac{\varphi}{2}$$

$$= \cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2} - 2\sin^2\frac{\varphi}{2}$$

$$= \cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}$$

$$= \cos\left(\frac{\varphi}{2} + \frac{\varphi}{2}\right) = \cos\varphi. \qquad \square$$

<u>geometr. Bedeutung in $\mathbb{C}^n$:</u>

$$\mathbb{C}^n \ni z = (z_1, \dots, z_n) \cong (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n) \in \mathbb{R}^{2n}$$

$$\|z\| = \sqrt{\sum_{i=1}^n \bar{z}_i z_i} = \sqrt{\sum_{i=1}^n |z_i|^2}$$

$$= \sqrt{\sum_{i=1}^n \left[(\operatorname{Re} z_i)^2 + (\operatorname{Im} z_i)^2\right]}$$

$$= \|(\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n)\|$$