Anwendung DGL

Hatten AWP $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ $\quad x: \mathbb{R} \to \mathbb{C}^n$ ges. $\quad A \in M(n, \mathbb{C})$ geg. $\quad x_0 \in \mathbb{C}^n$ geg.

hat die Lsg.

$$x(t) = e^{tA} x_0,$$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

Funktionalgl. Falls $AB = BA$, dann

$$e^{A+B} = e^A e^B.$$

Bew (bis auf Konvergenzfragen)

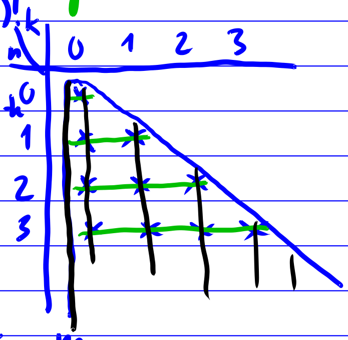
$$e^{A+B} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{(A+B)^n}$$

[binomischer Lehrsatz]

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\frac{n!}{k!(n-k)!}} A^k B^{n-k}$$

falls $AB = BA$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} A^k B^{n-k}$$



$[m = n-k]$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!\, m!} A^k B^m$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k \sum_{m=0}^{\infty} \frac{1}{m!} B^m$$

$$= e^A \, e^B \qquad \square$$

Praktische Berechnung von e^{tA}

$$A = S \; JNF(A) \; S^{-1}, \quad S \text{ bek.}$$

JNF(A) bek.

$$\Rightarrow e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{\left(S \; JNF(A) \; S^{-1}\right)^k}_{S \; JNF(A)^k \; S^{-1}}$$

$$= S \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} JNF(A)^k \right) S^{-1}$$

$$= S \; e^{t \, JNF(A)} \; S^{-1}$$

$$B \in Bl(n_1 \ldots n_r) \Rightarrow B^k \in Bl(n_1 \ldots n_r)$$

$$\Rightarrow e^B \in Bl(n_1 \ldots n_r)$$

JK $\quad J = \lambda E + N, \in M(m, \mathbb{C})$

$$\boxed{\begin{aligned}(t\lambda E)(tN) &= (tN)(t\lambda E)\\ &= t^2 \lambda N\end{aligned}}$$

$$e^{tJ} = e^{t\lambda E + tN}$$

$$= e^{t\lambda E} \, e^{tN}$$

$$= e^{t\lambda} E \, e^{tN}$$

$$= e^{t\lambda} e^{tN}$$

$$N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad N \in M(m, \mathbb{C})$$

$$N^m = 0 \implies e^{tN} = \sum_{k=0}^{m-1} \frac{1}{k!} t^k N^k$$

$$= \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{6}t^3 & & & & & \frac{t^{m-1}}{(m-1)!} \\ & 1 & t & \frac{1}{2}t^2 & \ddots & & & & \\ & & 1 & t & \frac{1}{2}t^2 & \ddots & & & \\ & & & 1 & t & \ddots & \ddots & & \\ & & & & 1 & t & \ddots & \ddots & \\ & & & & & 1 & \ddots & \ddots & \\ & & & & & & t & \frac{1}{2}t^2 & \frac{1}{6}t^3 \\ & 0 & & & & & 1 & t & \frac{1}{2}t^2 \\ & & & & & & & 1 & t \\ & & & & & & & & 1 \end{bmatrix}$$

Bsp

$$\exp t \left[\begin{array}{ccc|c} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ \hline & & & 2 \end{array}\right]$$

$$= e^{2t} \left[\begin{array}{ccc|c} 1 & t & \frac{t^2}{2} & \\ & 1 & t & \\ & & 1 & \\ \hline & & & 1 \end{array}\right].$$