Universität Tübingen, Mathematisches Institut Sommersemester 2021
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Groups and Representations

Instruction 1 for the preparation of the lecture on 21 April 2021

1.2 Basic notions[1]

Definition: (group)
Let $G \neq \emptyset$ be a set and let $\circ$ be an operation $\circ : G \times G \to G$. We call $(G, \circ)$ a group if:

(G1) $a, b \in G \Rightarrow a \circ b \in G$ (closure) (already implied by $\circ : G \times G \to G$)

(G2) $(a \circ b) \circ c = a \circ (b \circ c)\ \forall\ a, b, c \in G$ (associativity)

(G3) $\exists\ e \in G$ with $a \circ e = a = e \circ a\ \forall\ a \in G$ (identity/neutral element)

(G4) for each $a \in G\ \exists\ a^{-1} \in G$ with $a \circ a^{-1} = e = a^{-1} \circ a$, with e from (G3) (inverses)

Definition: (abelian group)
A group $(G, \circ)$ is called commutative or abelian, if in addition we have:
(G5) $a \circ b = b \circ a\ \forall\ a, b \in G$ (commutativity)

Remarks:

1. The identity e is unique. For each $a \in G$ the corresponding inverse is unique.
   Can you show this?
2. We often call the operation multiplication and write $a \cdot b$ or just ab instead of $a \circ b$.
3. If the number of group elements is finite, we speak of a *finite group*, and we call the number of group elements the *order* $|G|$ of the group. (otherwise: *infinite group*).
4. A finite group is completely determined by its *group table*:

   https://youtu.be/gmTSAOSs9U0 (4 min) (1)

   No two elements within one row (or column) can be the same. (see exercises)
   This implies the *rearrangement lemma*: If we multiply all elements of a group $\{e, a, b, c, \ldots\}$ by one of the elements, we obtains again all elements, in general in a different order.

Examples:

trivial group, $(\mathbb{Z}, +)$, and $(\mathbb{R} \setminus \{0\}, \cdot)$ https://youtu.be/FuUWrnBVstQ (2 min) (2)

symmetry group of an object https://youtu.be/olOM_fzkObA (4 min) (3)

Definition: (subgroup)
Let $(G, \circ)$ be a group. A subset $H \subseteq G$, which satisfies (G1)–(G4) (with the same operation $\circ$), is called a subgroup of G.

[1] section numbering according to lecture notes.

Remarks:

1. Every group has two trivial subgroups: $\{e\}$ and G.
   All other subgroups are called non-trivial.
2. If G is finite then $|H|$ divides $|G|$. (proof later)

Definition: (homomorphism)
Given two groups $(G, \circ)$ and $(G', \bullet)$, a map $f : G \to G'$ is called a homomorphism, if

$$f(a \circ b) = f(a) \bullet f(b) \quad \forall\, a, b \in G\,.$$

Remarks:

1. A homomorphism f maps the identity to the identity and inverses to inverses, more precisely $f(e_G) = e_{G'}$ and $f(a^{-1}) = f(a)^{-1}\ \forall\, a \in G$.
   Can you show this?
2. The *image* of the homomorphism $f : G \to G'$ is

$$\mathrm{im}(f) = f(G) = \{f(g)\ :\ g \in G\}\,,$$

   the *kernel* of f is the preimage of the identity of G' ,

$$\ker(f) = \{g \in G\ :\ f(g) = e_{G'}\}\,.$$

Definition: (isomorphism)
A bijective homomorphism $f : G \to G'$ is called isomorphism. We then say that G and G' are isomorphic, and write $G \cong G'$.

Remark: Isomorphic groups have the same group table, i.e. they are identical except for what we call their elements (and the group operation). (similarly for infinite groups)

1.3 Examples and outlook

Up to isomorphy there is only one group with two elements – but it comes in many guises:

https://youtu.be/7e8SMpY4Fk4 (9 min) (4)

Often we encounter cyclic groups:

https://youtu.be/TwITm0aX1gA (2 min) (5)

Functions that transform in a special way under a group will provide an interesting playing field for groups:

https://youtu.be/UdH9UIU5UCY (6 min) (6)