Universität Tübingen, Mathematisches Institut
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 2 for the preparation of the lecture on 26 April 2021

1.4 Permutations – the symmetric group

Definition: (symmetric group)
The symmetric group of degree n, S_n, are the bijections of $\{1, 2, \ldots, n\}$ to itself under composition.

Remarks:

1. Elements of S_n are called permutations.
2. $|S_n| = n!$

We use three notations for permutations:

two-line notation `https://youtu.be/0mjbR0pjkFs` (1 min) (1)

cycle notation `https://youtu.be/kvISarU6UWA` (5 min) (2)

birdtrack notation `https://youtu.be/1hllM7IPf3M` (4 min) (3)

Examples:

1. $S_2 = \{e, (12)\} \cong \mathbb{Z}_2$
2. $S_3 = \{e, (12), (13), (23), (123), (132)\}$
   Construct the group table! Is S_3 abelian?
   subgroups: $\{e\}$ and S_3 (trivial)
   $\{e, (12)\}$, $\{e, (13)\}$, $\{e, (23)\}$, all $\cong \mathbb{Z}_2$
   $\{e, (123), (321)\} \cong C_3$

Theorem 1. (Cayley)
Every group of order n is isomorphic to a subgroup of S_n.

Proof: `https://youtu.be/r4_oD2o6aqo` (5 min) (4)

Fun exercise (optional): Watch the video *An Impossible Bet* by `minutephysics`,

`https://youtu.be/eivGlBKlK6M` (2 min) (5)

and come up with a good strategy. Don't watch the solution! Think about cycles instead.

1.5 Group actions

Definition: (group action)
Let G be a group and M a set. A (group) action of G on M is a map

$$\begin{aligned} G \times M &\to M \\ (g, m) &\mapsto gm\,, \end{aligned}$$

which satisfies

$$\begin{aligned} em = m \quad &\forall\, m \in M \qquad \text{and} \\ g(hm) = (gh)m \quad &\forall\, g, h \in G \text{ and } \forall\, m \in M\,. \end{aligned}$$

Remark: Thus, $M \to M$, $m \mapsto gm$, is bijective for each (fixed) $g \in G$.
Can you show this?

Definition: (orbit)
The orbit of the point $m \in M$ under an action of a group G on M is defined as

$$Gm = \{gm \,:\, g \in G\}\,.$$

Remarks:

1. The orbit of a "typical" point contains $n = |G|$ elements.
2. The orbit of a "special" point contains less than $n = |G|$ elements.

Example: equilateral triangle https://youtu.be/1rUaIp5sJr8 (4 min) (6)

Definition: (stabiliser)
Let $G \times M \to M$, $(g, m) \mapsto gm$, be an action of G on M. The set of group elements that map a given $m \in M$ to itself, i.e.

$$G_m = \{g \in G \,:\, gm = m\}\,,$$

is called stabiliser (or isotropy group or little group) of m.

Remark: G_m is a group. (see exercises)

Example: equilateral triangle https://youtu.be/gPot13SMf0o (1 min) (7)

Notice that in all three cases $|Gm| \cdot |G_m| = |G|$. This is true in general for finite groups (*orbit-stabiliser theorem*, see exercises).

1.6 Conjugacy classes and normal subgroups

Definition: (conjugation)
Let G be a group. We say $x \in G$ is conjugate to $y \in G \underset{\text{Def.}}{\Leftrightarrow} \exists\, g \in G \,:\, y = gxg^{-1}$.
We then write $x \sim y$.

Show that $\sim$ is an equivalence relation, i.e. show reflexivity, symmetry and transitivity.

Examples: S_3, SO(3) https://youtu.be/LpBfagD302Q (6 min) (8)

Definition: (conjugacy class)
For a group G and $x \in G$ we call $\{gxg^{-1} \,:\, g \in G\}$ the conjugacy class of x.

Remarks:

1. The class of e contains only e, since $geg^{-1} = e \;\forall\, g$.
2. For abelian groups each element forms a class of its own, since $gxg^{-1} = x \;\forall\, g$.
3. In general a class is not a subgroup (cf. below).
4. Each element of G is contained in exactly one class. **Why?**
5. $|G|$ is divisible by the number of elements of each conjugacy class.
   (orbit-stabiliser theorem, see exercises)
6. Later: The number of conjugacy classes is equal to the number of non-equivalent irreducible representations of a finite group.

Example: conjugacy classes of S_3 https://youtu.be/F0r3dReVKCk (3 min) (9)

Definition: (conjugate subgroups, normal subgroup)

(i) We call a subgroup $K \subseteq G$ conjugate to a subgroup $H \subseteq G$ if $\exists\, g \in G$ such that

$$K = gHg^{-1} = \{ghg^{-1} \,:\, h \in H\}\,.$$

(ii) If $ghg^{-1} \in H \;\forall\, h \in H$ und $\forall\, g \in G$ then we call H a normal subgroup (or invariant subgroup) of G.

Study the behaviour of the subgroups of S_3 under conjugation!

Remark: A finite group is called *simple* if it has no non-trivial normal subgroup.