Universität Tübingen, Mathematisches Institut
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Sommersemester 2021

Groups and Representations

Instruction 3 for the preparation of the lecture on 28 April 2021

1.7 Cosets and quotient groups

Definition: (coset)
Let G be a group and $H \subseteq G$ a subgroup. For $g \in G$ the set

$$gH = \{gh : h \in H\}$$

is called a left coset of H (in G). Similarly, we call $Hg = \{hg : h \in H\}$ a right coset of H.

Remarks:

1. $gH, Hg \subseteq G$.
2. If $g \in H \Rightarrow gH = Hg = H$. **Why?**
3. $|gH| = |H|$. **Why?**
4. In the following we consider mostly left cosets.
5. Two cosets $g_1 H$ and $g_2 H$ are either identical ($\Leftrightarrow g_1^{-1} g_2 \in H$) or disjoint.

   Proof: https://youtu.be/yJI8R1ju87g (2 min) (1)

6. **Can you see** that this implies that $|H|$ divides $|G|$?

Example:

S_3 and subgroups
$H_1 = \{e, (12)\}$ (not normal) and
$H_2 = \{e, (123), (132)\}$ (normal)

https://youtu.be/SIA9F0k1JKQ (10 min) (2)

Definition: (quotient group)
Let H be a normal subgroup of G. We define the quotient group $(G/H, \cdot)$ as the set of cosets,

$$G/H = \{gH \ : \ g \in G\}\,, \qquad \text{with group law} \qquad (g_1 H) \cdot (g_2 H) = (g_1 g_2) H\,.$$

Remarks:

1. $|G/H| = \frac{|G|}{|H|}$
2. $(G/H, \cdot)$ is actually a group:

   https://youtu.be/N75Wz4j_Aa8 (3 min) (3)

   Where did we need that H is normal?

Example:

https://youtu.be/GodmqXXT-pM (1 min) (4)

1.8 Direct product

Definition: (direct product)
For two groups $(A, \circ)$ and $(B, \bullet)$ the direct product is the Cartesian product $A \times B$ with group law

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \circ a_2, b_1 \bullet b_2) \,.$$

Remarks:

1. $e_{A\times B} = (e_A, e_B)$ and $(a, b)^{-1} = (a^{-1}, b^{-1})$
2. for finite groups: $|A \times B| = |A||B|$
3. $G = A \times B$ has a normal subgroup isomorphic to A, and $G/A \cong B$ (and vice versa):

https://youtu.be/0ppFs0QuI9w (5 min) (5)

Caveat: In general, for a normal subgroup H of G, $G \ncong H \times (G/H)$. **Why not?**

1.9 Example: The homomorphism from SL(2, ℂ) to the Lorentz group

Let M be the Minkowski space, i.e. $M = \mathbb{R}^4$ with the Lorentz metric[1]

$$\|x\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \,.$$

We call $x = (x_0, x_1, x_2, x_3)$ a four-vector. A (homogeneous) Lorentz transformation Λ is a linear map $M \to M$, which preserves the Lorentz metric, i.e.

$$\|\Lambda x\|^2 = \|x\|^2 \qquad \forall\, x \in M \,.$$

The Lorentz group $L = \mathrm{O}(3, 1)$ is the group of all (homogeneous) Lorentz transformations. We identify each $x \in M$ with a Hermitian 2×2 matrix:

$$X = x_0 \mathbb{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \,, \quad \text{where}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

$$\text{i.e.} \quad X = \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} .$$

The σ_j are called Pauli matrices. **Convince yourself** that $\det X = \|x\|^2$.

Let's define a homomorphism from $\mathrm{SL}(2, \mathbb{C})$ to $L = O(3, 1)$:

https://youtu.be/GRIdoIQWCVg (5 min) (6)

The homomorphism $\phi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{O}(3, 1)$ is not an ismomorphism.

- **Show** that ϕ is not injective.
- ϕ is not surjective either:

https://youtu.be/YGpTOZ7cwXU (2 min) (7)

[1] More precisely, $\|x\|^2 = d(x, x)$ with the pseudo-Riemannian metric $d(x, y) = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$.