Universität Tübingen, Mathematisches Institut

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Groups and Representations

Instruction 4 for the preparation of the lecture on 3 May 2021

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2 Representations

2.1 Definitions

Definition: (representation)
Let G be a group and V a vector space. A representation (rep) Γ of G is a homomorphism $G \to \mathrm{GL}(V)$, i.e. into the bijective linear maps $V \to V$, i.e. in particular

$$\Gamma(g)\Gamma(h) = \Gamma(gh) \quad \forall\, g, h \in G$$

and $\Gamma(e) = \mathbb{1}$ (identity matrix/operator). We call $\dim V$ the dimension of the representation, and we will require $\dim V > 0$.

Remarks:

1. A representation is an action of G on V (in addition: linear).
2. We say that V *carries* the representation Γ, and we call V the *carrier space* (of Γ).
3. Unless otherwise stated we consider vector spaces over $\mathbb{C}$ (maybe sometimes over $\mathbb{R}$, probably never over other fields), e.g. $\mathbb{C}^n$ or $L^2(\mathbb{R}^d)$,[1] equipped with a scalar product $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$.
4. Choosing an orthonormal basis of V (if finite-dimensional), $\{v_j : j = 1, \dots, d = \dim V\}$, each $\Gamma(g)$ corresponds to a $d \times d$ matrix with elements

   $$\Gamma(g)_{jk} = \langle v_j | \Gamma(g) v_k \rangle \,,$$

   and we call Γ a *matrix representation*.

   We say: The v_i transform under G in the representation Γ.
5. $\dim V = \operatorname{tr} \Gamma(e)$ (if V is finite-dimensional)

Example:

a 3-dimensional rep of S_3 https://youtu.be/K2Dt1BGL1Vk (2 min) (1)

Determine $\Gamma(\underline{\times})$ and $\Gamma(\divideontimes)$.

Definition: (faithful representation)
We call a representation faithful if the homomorphism $\Gamma : G \to \mathrm{GL}(V)$ is injective, i.e. if different group elements are represented by different matrices.

[1] It's best to think of the finite-dimensional case for the moment. In the infinite-dimensional case we'd really want separable Hilbert spaces and bounded linear operators $\Gamma(g)$.

Remarks:

1. Every group has the trivial representation, with $\Gamma(g) = \mathbb{1} \ \forall\, g \in G$; in general not faithful.
2. If G has a non-trivial normal subgroup H, then a representation of the quotient group G/H induces a representation of G. This representation is not faithful.

   https://youtu.be/PS2YTz14a2Y (3 min) (2)

   Show: If a non-trivial rep Γ is not faithful, then G has a non-trivial normal subgroup H, and Γ induces a faithful representation of the quotient group G/H.

Definition: (unitary representation)
A representation $\Gamma : G \to \mathrm{GL}(V)$ is called unitary, if $\Gamma(g)$ is unitary $\forall\, g \in G$, i.e. $\langle \Gamma(g)v|\Gamma(g)w\rangle = \langle v|w\rangle \ \forall\, v, w \in V$.

Remarks:

1. If V is finite-dimensional and if we choose an orthonormal basis, then such a representation is in terms of unitary matrices.
2. Unitary representations are important for applications in physics, since it is in terms of them that symmetries are implemented in quantum mechanics (or quantum field theory).

2.2 Equivalent Representations

Definition: (equivalent representations)
We say that two representations $\Gamma : G \to \mathrm{GL}(V)$ and $\tilde{\Gamma} : G \to \mathrm{GL}(W)$ are equivalent, if there exists an invertible linear map $S : V \to W$ such that

$$\Gamma(g) = S^{-1}\,\tilde{\Gamma}(g)\,S \quad \forall\, g \in G\,.$$

Remark: If the linear map is unitary, i.e. (writing U instead of S) $U : V \to W$ with $\langle U\phi|U\psi\rangle_W = \langle \phi|\psi\rangle_V$ then we say that the representations are *unitarily equivalent.* For finite-dimensional representations we have $V \cong W \cong \mathbb{C}^{\dim V}$, and by choosing orthonormal bases U becomes a unitary matrix.

Theorem 2. *Let G be a finite group, $\Gamma : G \to \mathrm{GL}(V)$ a (finite-dimensional) representation and $\langle\cdot|\cdot\rangle$ a scalar product on V. Then Γ is equivalent to a unitary representation.*

Proof: $(v, w) = \sum_{g\in G} \langle \Gamma(g)v|\Gamma(g)w\rangle$ is also a scalar product:

https://youtu.be/-HWa-iaBZVk (4 min) (3)

Let $\{v_j\}$ be an orthonormal basis (ONB) with respect to $\langle\cdot|\cdot\rangle$ and $\{w_j\}$ an ONB with respect to $(\cdot,\cdot)$. Then there exists an invertible map $S : V \to V$ with $Sw_j = v_j$ (change of basis). Hence

$(v, w) = \langle Sv|Sw\rangle\,.$ https://youtu.be/L_HIR-Ug7nc (4 min) (4)

Finally, $\tilde{\Gamma}$ with $\tilde{\Gamma}(g) = S\Gamma(g)S^{-1}$ is equivalent to Γ and unitary:

https://youtu.be/1iXbQXyYvjY (6 min) (5)

2.4 Irreducible Representations

Definition: (invariant subspace)
Let $\Gamma : G \to \mathrm{GL}(V)$ be a representation and $U \subseteq V$ a subspace of V. U is called invariant subspace (with respect to Γ), if $\Gamma(g)v \in U \;\forall\, v \in U$ and $\forall\, g \in G$.

Remark: Every carrier space has two trivial invariant subspaces, namely V and $\{0\}$. All other invariant subspace (if there are any) are called non-trivial.

Definition: (irreducible representation & complete reducibility)

We call a representation $\Gamma : G \to \mathrm{GL}(V)$

(i) irreducible, if V possesses no non-trivial invariant subspace. Then we also call V irreducible with respect to Γ.

(ii) reducible, if V possesses a non-trivial invariant subspace U.

(iii) completely reducible, if V can be written as a direct sum of irreducible invariant subspaces.

Abbreviation for "irreducible representation": *irrep*

Example:

https://youtu.be/kfpZhLGZ9IA (5 min) (6)

Write $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ as linear combination of vectors from U_1 and U_2. **Construct** an ONB (with respect to the canonical scalar product) s.t. the first basis vector spans U_1 and the other two span U_2.