Universität Tübingen, Mathematisches Institut
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 5 for the preparation of the lecture on 5 May 2021

2.4 Irreducible Representations (cont.)

Theorem 3. *Let $\Gamma : G \to \mathrm{GL}(V)$ be a unitary representation and $U \subseteq V$ an invariant subspace. Then:*

(i) $U^\perp = \{v \in V \,:\, \langle u|v\rangle = 0 \quad \forall\, u \in U\}$ *is also invariant,*

(ii) *the restrictions* $\Gamma|_U$ *and* $\Gamma|_{U^\perp}$ *define representations* Γ^1 *and* Γ^2*, and*

(iii) Γ *ist equivalent to* $\Gamma^1 \oplus \Gamma^2$*; we simply write* $\Gamma = \Gamma^1 \oplus \Gamma^2$*.*

Corollary: (Maschke's Theorem)
We can write every (finite-dimensional) unitary representation as a direct sum of irreducible representations.

Explain why this implies that for finite groups every (finite-dimensional) representation is completely reducible.

Proof (of Theorem 3 & Corollary):

https://youtu.be/FJjdh6WNVF8 (4 min) (1)

Remark: Given a completely reducible representation $\Gamma : G \to \mathrm{GL}(V)$, we can find a basis of V such that in matrix notation

$$\Gamma(g) = \begin{pmatrix} \Gamma^1(g) & & & \mathbf{0} \\ & \Gamma^2(g) & & \\ & & \Gamma^3(g) & \\ \mathbf{0} & & & \ddots \end{pmatrix},$$

where the Γ^j are irreducible ($d_j \times d_j$ blocks with $d_j = \dim \Gamma^j$).

Here an irreducible representation can appear more than once, (relabel)

$$\Gamma = \underbrace{\Gamma^1 \oplus \cdots \oplus \Gamma^1}_{a_1 \text{ times}} \oplus \underbrace{\Gamma^2 \oplus \cdots \oplus \Gamma^2}_{a_2 \text{ times}} \oplus \cdots = \bigoplus_j a_j \Gamma^j\,,$$

https://youtu.be/f-9KAeE3oPc (2 min)

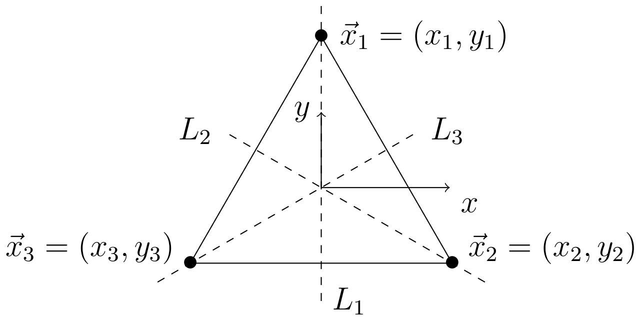
i.e. in Γ the irreducible representation Γ^j is contained a_j times.

2.4.1 Example: O_A operators for the group D_3

Consider a group G of orthogonal matrices $A : \mathbb{R}^n \to \mathbb{R}^n$ and some functions $\varphi : \mathbb{R}^n \to \mathbb{C}$. Then $(O_A\varphi)(\vec{x}) = \varphi(A^{-1}\vec{x})$ defines a representation of G on some function space.

https://youtu.be/9qYex-CZTf0 (2 min) (2)

Choose $G = D_3$, the symmetry group of an equilateral triangle ($\cong S_3$),



$n = 2$, and $\varphi_j(\vec{x}) = \mathrm{e}^{-|\vec{x}-\vec{x}_j|^2}$. Then $\mathrm{span}(\varphi_1, \varphi_2, \varphi_3)$ is invariant and carries a three-dimensional rep of S_3:

https://youtu.be/v3hSAc-h7mg (5 min) (3)

We find two invariant subspaces, one carries the trivial rep, and the other carries a two-dimensional rep:

https://youtu.be/FdpzE7YqR_k (4 min) (4)

2.5 Schur's Lemmas and orthogonality of irreps

Theorem 4. (Schur's Lemma 1)
Let G be a group, $\Gamma : G \to \mathrm{GL}(V)$ a finite-dimensional, irreducible representation and $A : V \to V$ a linear map. If A commutes with Γ, i.e. $A\Gamma(g) = \Gamma(g)A \ \forall\, g \in G$, then $A = c\mathbb{1}$ for some $c \in \mathbb{C}$.

Proof: https://youtu.be/JStDaicTKaU (3 min) (5)

Corollary: *For an abelian group G, every irreducible representation has dimension* 1. **Explain!**

Theorem 5. (Schur's Lemma 2)
Let G be a group, $\Gamma : G \to \mathrm{GL}(V)$ and $\tilde{\Gamma} : G \to \mathrm{GL}(W)$ two finite-dimensional, unitary irreducible representations and $A : V \to W$ a linear map. If

$$A\Gamma(g) = \tilde{\Gamma}(g)A \quad \forall\, g \in G\,,$$

then $A = 0$ or Γ and $\tilde{\Gamma}$ are unitarily equivalent.

Proof: https://youtu.be/Or9gnVwmuDo (5 min) (6)