Universität Tübingen, Mathematisches Institut
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 6 for the preparation of the lecture on 10 May 2021

2.5 Schur's Lemmas and orthogonality of irreps (cont.)

Theorem 6. *Let G be a finite group and Γ^j, $j = 1, 2, \ldots$, non-equivalent unitary irreducible representations with $\dim \Gamma^j = d_j$. Then the matrix elements obey the orthogonality relation*

$$\frac{1}{|G|} \sum_{g \in G} \overline{(\Gamma^j(g)_{\mu\nu})}\, \Gamma^k(g)_{\mu'\nu'} = \frac{1}{d_j} \delta_{jk}\, \delta_{\mu\mu'}\, \delta_{\nu\nu'}$$

$\forall\ \mu, \nu = 1, \ldots, d_j$ *and* $\forall\ \mu', \nu' = 1, \ldots, d_k$.

Proof: https://youtu.be/vWhHL-2cCTw (13 min) (1)

Corollary 1 to Theorem 6:

$$\sum_j d_j^2 \leq |G|$$ https://youtu.be/bHY8dAFQA-c (4 min) (2)

Remark: Later we will see that we actually have equality.

2.6 Characters

Definition: (character)
For a finite-dimensional representation $\Gamma : G \to \mathrm{GL}(V)$ we call $\chi : G \to \mathbb{C}$ with

$$\chi(g) = \operatorname{tr} \Gamma(g)$$

the character of the representation.

Remarks:

1. In terms of matrix elements we have $\chi(g) = \sum_{\mu=1}^{\dim V} \Gamma(g)_{\mu\mu}$.
2. Equivalent reps have the same characters.
3. Characters are constant on conjugacy classes.

Show remarks 2 and 3.

Corollary 2 to Theorem 6. *Let G be a finite group and Γ^j, $j = 1, 2, \ldots$, non-equivalent, irreducible representations with $\dim \Gamma^j = d_j$. Then the characters $\chi^j = \operatorname{tr} \Gamma^j$ obey the orthogonality relation*

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi^j(g)}\, \chi^k(g) = \delta_{jk}\,.$$

Proof: https://youtu.be/q1P7YKGsFWg (2 min) (3)

Remarks:

1. Since the characters depend only on the conjugacy class, we can rewrite the orthogonality relation as

$$\frac{1}{|G|}\sum_c n_c\,\overline{\chi_c^j}\,\chi_c^k = \delta_{jk}\,,$$

where c labels classes and n_c is the number of elements in class c.

2. Let m be the number of different conjugacy classes of G, and let p the number of non-equivalent irreducible representations. Then

$$p \leq m\,. \qquad \texttt{https://youtu.be/60Muqu1iMNk}\ (3\,\text{min})$$

In the exercises you will show that we actually have $p = m$.
The $m \times m$ matrix with entries χ_c^j is called *character table* of the group.

3. If Γ is irreducible then

$$\frac{1}{|G|}\sum_{g\in G} |\chi(g)|^2 = 1\,.$$

If Γ is reducible then

$$\frac{1}{|G|}\sum_{g\in G} |\chi(g)|^2 > 1\,.$$

https://youtu.be/aewkyIA009c (4 min) (4)

In https://youtu.be/FdpzE7YqR_k we encountered three reps of S_3.
Check for irreducibility!

Example: Here's another irrep of $D_3 \cong S_3$:

$$\Gamma(e) = \Gamma(C) = \Gamma(\bar{C}) = 1\,, \qquad \Gamma(\sigma_1) = \Gamma(\sigma_2) = \Gamma(\sigma_3) = -1$$

Hence, the character table of $D_3 \cong S_3$ reads

		$\{e\}$	$\{C,\bar{C}\}$	$\{\sigma_1,\sigma_2,\sigma_3\}$
trivial rep	χ^1	1	1	1
other 1D rep	χ^2	1	1	-1
2D irrep	χ^3	2	-1	0

Remarks:

4. If $\Gamma = \bigoplus_j a_j \Gamma^j$ with irreps Γ^j then

$$a_j = \frac{1}{|G|}\sum_c n_c\,\overline{\chi_c^j}\,\chi_c\,. \qquad \texttt{https://youtu.be/yW5um6Cl0k4}\ (2\,\text{min}) \qquad (5)$$

Use this in order to verify that the 3D rep of $D_3 \cong S_3$ from https://youtu.be/FdpzE7YqR_k is a direct sum of the trivial rep of the 2D irrep.

Supplement: D_3-reps from https://youtu.be/FdpzE7YqR_k.
3D rep:

$$\Gamma(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma(C) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Gamma(\bar{C}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Gamma(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Gamma(\sigma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Gamma(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2D irrep:

$$\Gamma^3(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^3(C) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \Gamma^3(\bar{C}) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\Gamma^3(\sigma_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^3(\sigma_2) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \Gamma^3(\sigma_3) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$