Universität Tübingen, Mathematisches Institut
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 7 for the preparation of the lecture on 12 May 2021

2.7 The regular representation

Definition: (group algebra)
For a finite group G we define its group algebra $\mathcal{A}(G)$ as the vector space spanned by the group elements, i.e. we form linear combinations

$$\mathcal{A}(G) \ \ni \ r = \sum_{j=1}^{|G|} r_j g_j \,, \quad r_j \in \mathbb{C} \,,$$

with multiplication rule

$$\left(\sum_{j=1}^{|G|} q_j g_j \right) \left(\sum_{k=1}^{|G|} r_k g_k \right) = \sum_{j=1}^{|G|} \sum_{k=1}^{|G|} q_j r_k \, g_j g_k \,.$$

induced by group multiplication.

Remarks:

https://youtu.be/3QjW40hhVag (3 min) (1)

Now we can write group multiplication as

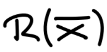
$$g g_j = \sum_{k=1}^{|G|} g_k \, R(g)_{kj} \,,$$

where $R(g)_{kj}$ encodes the group table: For g and j fixed, $R(g)_{kj} = 1$ for exactly one value of k and it vanishes for all others. R defines a representation of G on $\mathcal{A}(G)$, the so-called *regular representation*:

https://youtu.be/xgF-ifx0sgg (4 min) (2)

Example:

regular rep R of S_3 https://youtu.be/Hbd2obSIQ1g (5 min) (3)

Determine  with the same choice of basis as in the video.

Theorem 7. *The regular representation R contains all irreps of G, and the multiplicity of irrep Γ^j is given by its dimension d_j,*

$$R = \bigoplus_{j=1}^{p} d_j \, \Gamma^j \qquad \left(p \ = \ \begin{array}{c} \textit{number of non-equivalent} \\ \textit{irreducible representations} \end{array} \right) \,.$$

Remark: Hence, there exists a regular matrix S such that

$$S^{-1}\,R(g)\,S = \begin{pmatrix} 1 & & & & & & \\ & \Gamma^2(g) & & & & & \\ & & \ddots & & & & \\ & & & \Gamma^2(g) & & & \\ & & & & \ddots & & \\ & & & & & \Gamma^p(g) & \\ & & & & & & \ddots \\ & & & & & & & \Gamma^p(g) \end{pmatrix} .$$

$\underbrace{\qquad\qquad}_{d_2 \text{ blocks}} \cdots \underbrace{\qquad\qquad}_{d_p \text{ blocks}}$

Proof:

https://youtu.be/-QPH0cyBLpc (4 min) (4)

Corollary. *We have* $\sum_{j=1}^{p} d_j^2 = |G|$.

Show this!

2.8 Product representations and Clebsch-Gordan coefficients

Definition: (product representation)
For representations $\Gamma^\mu : G \to \mathrm{GL}(U)$ and $\Gamma^\nu : G \to \mathrm{GL}(V)$ we define the product representation $\Gamma^{\mu\otimes\nu} : G \to \mathrm{GL}(U \otimes V)$ by

$$\Gamma^{\mu\otimes\nu}(g) = \Gamma^\mu(g) \otimes \Gamma^\nu(g) \quad \forall\, g \in G\,.$$

Remarks:

1. $\Gamma^{\mu\otimes\nu}$ is a representation:

   https://youtu.be/s8aENniin5Y (3 min) (5)

2. For the characters we have

$$\chi^{\mu\otimes\nu}(g) = \mathrm{tr}\,\Gamma^{\mu\otimes\nu}(g) = \mathrm{tr}\left(\Gamma^\mu(g)\otimes\Gamma^\nu(g)\right) = \mathrm{tr}\,\Gamma^\mu(g)\ \mathrm{tr}\,\Gamma^\nu(g) = \chi^\mu(g)\chi^\nu(g)\,.$$

3. Even for irreducible Γ^μ and Γ^ν the product representation is in general reducible,

$$\Gamma^\mu \otimes \Gamma^\nu = \bigoplus_\lambda a_\lambda \Gamma^\lambda \quad \text{with} \quad d_\mu d_\nu = \sum_\lambda a_\lambda d_\lambda\,.$$

   According to character orthogonality the multiplicities are

$$a_\lambda = \frac{1}{|G|}\sum_c n_c\,\overline{\chi_c^\lambda}\,\chi_c^\mu \chi_c^\nu\,.$$

Example: $\mathbb{Z}_2 \cong \{e, P\}$ has two one-dimensional irreps. Character table:

	e	P
$\chi^1 = \Gamma^1$	1	1
$\chi^2 = \Gamma^2$	1	-1

Construct the regular rep R.
Reduce $\Gamma = R \otimes R$ to a direct sum of irreps.

Clebsch-Gordan coefficients. In our live session we will go through some awkward looking but frequently used notation in the context of the basis change from a product basis to basis in which subsets of the basis vectors span irreducible subspaces.

Recap: Tensor products

Let U and V be vector spaces with bases $\{u_i\}$ and $\{v_j\}$, respectively, and let $W = U \otimes V$ with basis $\{w_k\}$, where $w_k = u_i \otimes v_j$. Further let $A : U \to U$ and $B : V \to V$ be linear maps. Then $D := A \otimes B$ is the linear map $W \to W$ with

$$Dw_k = Au_i \otimes Bv_j \,, \quad \text{where } k = (i, j) \,,$$

by linearity extended to arbitrary $w \in W$, i.e. for $w = \sum_k \alpha_k w_k = \sum_{ij} \alpha_{ij} u_i \otimes v_j$ we have

$$Dw = \sum_{i,j} \alpha_{ij} \, Au_i \otimes Bv_j \,.$$

In matrix components:

$$Au_i = \sum_{i'} u_{i'} A_{i'i} \,, \qquad Bv_j = \sum_{j'} v_{j'} B_{j'j} \quad \text{and}$$
$$Dw_k = \sum_{k'} w_{k'} D_{k'k} = \sum_{i'j'} (u_{i'} \otimes v_j) A_{i'i} \, B_{j'j} \,,$$

i.e. $D_{k'k} \equiv D_{i'j'ij} = A_{i'i} \, B_{j'j}$. If everything is finite-dimensional then

$$\operatorname{tr} D = \sum_k D_{kk} = \sum_{i,j} A_{ii} B_{jj} = \operatorname{tr} A \cdot \operatorname{tr} B = \operatorname{tr}(A \otimes B) \,.$$

Scalar products on U and V induce a scalar product on W by

$$\langle w_k | w_{k'} \rangle = \langle u_i | u_{i'} \rangle_U \, \langle v_j | v_{j'} \rangle_V \,,$$

again extended by (sesqui-)linearity.
If $\{u_i\}$ and $\{v_j\}$ are ONB with respect to $\langle \,|\, \rangle_U$ and $\langle \,|\, \rangle_V$, then $\{w_k\}$ is also orthonormal,

$$\langle w_k | w_{k'} \rangle = \delta_{ii'} \delta_{jj'} = \delta_{kk'} \,.$$