Universität Tübingen, Mathematisches Institut
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 9 for the preparation of the lecture on 19 May 2021

3.3 Perturbation theory and lifting of degeneracies

Setting: Hamiltonian is a sum of a (known) term H_0 and a (small) perturbation H',

$$H = H_0 + H' \,.$$

Let G be the symmetry group of H_0. Two possibilities:

1. H' is also invariant under G.
2. H' is only invariant under a subgroup $B \subset G$.

In case 1 the spectra of H_0 and of H look similar (same multiplicities).

Case 2 (symmetry breaking) typically leads to a splitting of energy levels:

- Eigenstates of H transform in irreps of B.
- Degenerate eigenstates of H_0 transform in irreps of G.
- Eigenspaces of H_0 carry reps of B, in general reducible.
  States transforming in different irreps of B, in general, have different energies.
  States transforming in the same irrep of B, are still degenerate.

https://youtu.be/_IDScHV5Jps (3 min) (1)

Examples:

1. Hydrogen atom as in Section 3.2.
   Adding a small radially symmetric potential $V(r)$ (but not $\frac{1}{r}$) breaks the O(4)-symmetry to O(3). Each energy level splits into n levels with different ℓ. Each new level is still $(2\ell{+}1)$-fold degenerate.

   https://youtu.be/y_tIHpehjcY (2 min) (2)

2. Fine structure of hydrogen.
   - Take electron spin into account: instead of $L^2(\mathbb{R}^3)$ consider $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.
   - Intermediate step: Consider $H \otimes \mathbb{1}_2$. States which so far transformed in irrep $\Gamma^{2\ell+1}$ of O(3), now transform in rep $\Gamma^{2\ell+1} \otimes \Gamma^2$ of SU(2), but energies are unchanged, only the degeneracy is doubled.

     Wait, why SU(2)*?* https://youtu.be/2dFq2LwrrMU (4 min) (3)

   - Now add the perturbation H', containing spin-dependent terms (spin-orbit coupling), but still invariant under SU(2). Splittings follow from

     $$\Gamma^{2\ell+1} \otimes \Gamma^2 = \Gamma^{2\ell} \oplus \Gamma^{2\ell+2} \,.$$

     https://youtu.be/p1SZsPfGjEM (7 min) (4)

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4 Expansion into irreducible basis vectors

4.1 Projection operators onto irreducible bases

Recall Lemma 8 and the following remark about constructing irreducible invariant subspaces. Let's elaborate on this idea. Let U be a (completely reducible) representation (e.g. by unitary operators) on V and let $e_1^\nu, \dots, e_{d_\nu}^\nu \in V$ be functions/vectors that transform in the unitary irreducible representation Γ^ν (with $\dim(\Gamma^\nu) = d_\nu$). We can expand every $\psi \in V$ into such basis vectors, i.e.

$$\psi = \sum_\mu \sum_{\beta=1}^{d_\mu} c_\beta^\mu \, e_\beta^\mu \, ,$$

with expansion coefficients $c_\beta^\mu \in \mathbb{C}$. Let's apply $U(g)$:

https://youtu.be/ZA1qsZNHl5M (6 min) (5)

This motivates the following definition.

Definition: (generalised projection operators)
Let G be a group, U a representation, Γ^μ an irreducible representation, $\dim \Gamma^\mu = d_\mu$. We call

$$P_{jk}^\mu = \frac{d_\mu}{|G|} \sum_{g \in G} [\Gamma^\mu(g)^{-1}]_{jk} \, U(g)$$

generalised projection operator.

Remark: In the following Γ will always be unitary, i.e.

$$[\Gamma^\mu(g)^{-1}]_{jk} = [\Gamma^\mu(g)^\dagger]_{jk} = \overline{\Gamma^\mu(g)_{kj}} \qquad \text{(cf. above)}.$$

We will study the properties of these operators on the next instruction sheet.