Universität Tübingen, Mathematisches Institut
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Sommersemester 2021

Groups and Representations

Instruction 10 for the preparation of the lecture on 31 May 2021

4.1 Projection operators onto irreducible bases (cont.)

Theorem 9. (Properties of P^μ_{jk}) *With the above definitions we have:*

(i) *For fixed $\psi \in V$ and for fixed μ and j the d_μ vectors $P^\mu_{jk}\psi$, $k = 1, \ldots, d_\mu$, either all vanish or they transform in irrep Γ^μ, i.e. $U(g)P^\mu_{jk} = \sum_\ell P^\mu_{j\ell}\Gamma^\nu(g)_{\ell k}$.*

(ii) $P^\mu_{ji}P^\nu_{\ell k} = \delta_{\mu\nu}\delta_{jk}P^\mu_{\ell i}$.

(iii) $P^\mu_j = P^\mu_{jj}$ *is a projection operator.*

(iv) $P^\mu = \sum_j P^\mu_j$ *is a projection operator onto the invariant subspace U^μ containing all vectors transforming in the irreducible representation Γ^μ.*

(v) $\sum_\mu P^\mu = \mathbb{1}$ *if V completely reducible (here always assumed).*

(vi) $U(g) = \sum_\mu \sum_{j,k} \Gamma^\mu(g)_{kj} P^\mu_{jk}$ *(inversion of definition).*

Proof:

(i) & (ii)	https://youtu.be/Xenr0VXpvcM (4 min)	(1)
(iii)–(v)	https://youtu.be/05OMW7Cao8w (2 min)	(2)
(vi)	https://youtu.be/M-4KmZHsM0w (2 min)	(3)

Examples:

1. Reduction of span(ϕ_1, ϕ_2, ϕ_3) from Section 2.4.1 (invariant under $D_3 \cong S_3$). S_3 has two 1-dimensional and one 2-dimensional irrep ($\Gamma^1, \Gamma^2, \Gamma^3$).

   generalised projection operators https://youtu.be/laouieOnL4A (6 min) (4)

   Apply to some vector, say ϕ_1:

$\mu = 1, 2$	https://youtu.be/nMMHx7_zs_w (3 min)	(5)
$\mu = 3$	https://youtu.be/8sDomkziGvA (5 min)	(6)

2. Reducing a product representation:

   https://youtu.be/79QuhXEDkGY (3 min) (7)

4.2 Irreducible operators and the Wigner-Eckart Theorem

Definition: (irreducible operators)
Let G be a group, $U : G \to \mathrm{GL}(V)$ a representation, and Γ^μ a unitary irreducible representation with $\dim \Gamma^\mu = d_\mu$. A set of linear operators $O_i^\mu : V \to V$, $i = 1, \ldots, d_\mu$, which transform under G as follows,

$$U(g)\, O_i^\mu\, U(g)^{-1} = \sum_{j=1}^{d_\mu} O_j^\mu\, \Gamma^\mu(g)_{ji}\,,$$

is called a set of irreducible operators corresponding to irrep Γ^μ.
(The O_i^μ are also called irreducible tensors or irreducible tensor operators).

Remarks:

1. The definition makes sense:

   https://youtu.be/KEr1n5iC394 (4 min) (8)

2. Special case: If Γ^μ is the trivial representation then the operator O^μ (no index i, since $d_\mu = 1$) commutes with $U(g)\ \forall\, g \in G$, cf. Section 3.2.

3. If O_i^μ, $i = 1, \ldots, d_\mu$, are irreducible operators and if $|e_j^\nu\rangle$, $j = 1, \ldots, d_\nu$, are irreducible basis vectors, then the vectors $O_i^\mu |e_j^\nu\rangle$ transform in the product rep $\Gamma^{\mu\otimes\nu}$.
   Show this!
   We can reduce this product representation (cf. Section 2.8) and expand the vectors $O_i^\mu |e_j^\nu\rangle$ in the irreducible basis $\{|w_{\alpha\ell}^\lambda\rangle\}$,

   $$O_i^\mu |e_j^\nu\rangle = \sum_{\alpha\lambda\ell} |w_{\alpha\ell}^\lambda\rangle \langle \alpha, \lambda, \ell(\mu,\nu) i, j\rangle\,. \qquad (*)$$

   This leads to...

Theorem 10. (Wigner-Eckart)
Let O_i^μ be irreducible operators and let $|e_j^\nu\rangle$ be irreducible vectors. Then

$$\langle e_\ell^\lambda | O_i^\mu | e_j^\nu \rangle = \sum_\alpha \langle \alpha, \lambda, \ell(\mu,\nu) i, j\rangle\, \langle \lambda \| O^\mu \| \nu \rangle_\alpha$$

with the so-called reduced matrix element (which isn't a matrix element...)

$$\langle \lambda \| O^\mu \| \nu \rangle_\alpha = \frac{1}{d_\lambda} \sum_k \langle e_k^\lambda | w_{\alpha k}^\lambda \rangle\,.$$

Can you prove this, using $(*)$ and the proof of Lemma 8?