Universität Tübingen, Mathematisches Institut Sommersemester 2021
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Groups and Representations

Instruction 11 for the preparation of the lecture on 2 June 2021

4.2 Irreducible operators and the Wigner-Eckart Theorem (cont.)

Remarks (on Wigner-Eckart):

https://youtu.be/DnBKfkmI4R8 (4 min) (1)

Example: time-dependent perturbation theory for dipole radiation

https://youtu.be/iTP7E2z54F8 (8 min) (2)

4.3 Left ideals and idempotents

The generalised projection operators allow us to decompose reducible reps into sums of irreps. To this end we already have to know the irreps.

Question: How to construct the irreps?

Idea: Reduce the regular rep (see Instruction 7), as it contains all irreps Γ^μ

Recall: Carrier space is the group algebra $\mathcal{A}(G) = \operatorname{span}(g_1, \dots, g_{|G|})$.

Definition: (left ideal, minimal left ideal)
A subspace $L \subseteq \mathcal{A}(G)$ that is invariant under left multiplication is called *left ideal*, i.e.

$$r \in L \text{ and } q \in \mathcal{A}(G) \quad \Rightarrow \quad qr \in L\,.$$

A left ideal L is called *minimal* if it does not contain any non-trivial left ideal $K \subset L$.

Remarks:

1. One similarly defines right ideals and two-sided ideals. (We use only left ideals.)
2. L is a left ideal $\Leftrightarrow$ L is an invariant subspace.
3. L is a minimal left ideal $\Leftrightarrow$ L is an irreducible invariant subspace.

Show remarks 2 and 3.

Denote by P^μ_α the projection operator onto the minimal left ideal L^μ_α, i.e. $P^\mu_\alpha \mathcal{A}(G) = L^\mu_\alpha$. As before μ labels the non-equivalent irreps, and $\alpha = 1, \dots, d_\mu$. Demand the following...

... properties of P^μ_α:

(i) $P^\mu_\alpha r \in L^\mu_\alpha \quad \forall\, r \in \mathcal{A}(G)$.

(ii) If $q \in L^\mu_\alpha$ then $P^\mu_\alpha q = q$.

(iii) $P^\mu_\alpha P^\nu_\beta = \delta_{\mu\nu}\delta_{\alpha\beta} P^\mu_\alpha$.

It then follows that

(iv) $P^\mu_\alpha q = q P^\mu_\alpha \quad \forall\, q \in \mathcal{A}(G)$.

Proof: https://youtu.be/ebET50quvPk (3 min) (3)

We define $L^\mu = \bigoplus_\alpha L^\mu_\alpha$ and first construct the projection operator P^μ onto L^μ:

https://youtu.be/YLY-j-LNkH8 (4 min) (4)

Lemma 11. *P^μ is given by right multiplication with e_μ, i.e. $P^\mu q = q e_\mu \ \forall\, q \in \mathcal{A}(G)$.*

Remarks:

1. P^μ is linear.
2. $e_\mu e_\nu = \delta_{\mu\nu} e_\mu$ – cf. property (iii). **Show this.**
3. With $e = \sum_{\mu,\alpha} e^\mu_\alpha$ this also works for projections to minimal left ideals, $P^\mu_\alpha q = q e^\mu_\alpha$.

Definition: (idempotents)
An element $e_\mu \in \mathcal{A}(G)$ that satisfies $e_\mu^2 = e_\mu$ is called (an) idempotent. If $e_\mu^2 = \xi_\mu e_\mu$ for some non-zero $\xi_\mu \in \mathbb{C}$ then we call e_μ essentially idempotent.

Remarks:

1. We say the idempotent e_μ *generates* the left ideal L^μ, i.e. $L^\mu = \{q e_\mu \,:\, q \in \mathcal{A}(G)\}$.
2. An idempotent is called *primitive*, if it generates a minimal left ideal. Otherwise it can be written as a sum $e_1 + e_2$ of two non-zero idempotents with $e_1 e_2 = 0 = e_2 e_1$.

Theorem 12.
The idempotent e_μ is primitive. $\Leftrightarrow$ For every $q \in \mathcal{A}(G)\ \exists\, \lambda_q \in \mathbb{C}$ s.t. $e_\mu q e_\mu = \lambda_q e_\mu$.

Proof:

https://youtu.be/jrqF23SpENg (10 min) (5)

Theorem 13. *The left ideals generated by two primitive idempotents, e_1 and e_2, carry equivalent irreps Γ^1 and Γ^2 iff $e_1 q e_2 \neq 0$ for at least one $q \in \mathcal{A}(G)$.*

Proof:

https://youtu.be/Wy3NS9IE_oY (9 min) (6)

Example: The primitive idempotent

$$e_1 = \frac{1}{|G|} \sum_{j=1}^{|G|} g_j$$

generates the one-dimensional left ideal L^1, which carries the trivial representation.
Show this!