Universität Tübingen, Mathematisches Institut
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 12 for the preparation of the lecture on 7 June 2021

4.3.1 Dimensions and characters of the irreducible representations

Theorem 14. *Let G be a group with group algebra $\mathcal{A}(G)$, and let*

$$e_\mu = \sum_{g\in G} a_g\, g$$

be a primitive idempotent with corresponding left ideal $L^\mu = \mathcal{A}(G)e_\mu$, carrying irrep Γ^μ with $\dim\Gamma^\mu = d_\mu$. Then $\forall\, h \in G$

$$\chi^\mu(h) = \operatorname{tr}\Gamma^\mu(h) = \frac{|G|}{n_c}\sum_{g\in c}\overline{a_g}$$

where c is the conjugacy class of h with n_c elements.

Proof:

https://youtu.be/iSiwLsOS8w8 (12 min) (1)

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5 Representations of the symmetric group and Young diagrams

5.1 One-dimensional irreps and associate representations of S_n

The *alternating group* A_n is the group of even permutations of $\{1, 2, \dots, n\}$ (i.e. each element is the product of an even number of transpositions). A_n is a normal subgroup of S_n, with quotient group $S_n/A_n \cong \mathbb{Z}_2$.

⇒ S_n has two one-dimensional representations, induced by the by the representations of $\mathbb{Z}_2$ (cf. Problems 8, 14 & 15):

$$\Gamma^{\mathrm{s}}(p) = 1 \quad \forall\, p \in S_n \text{ (trivial representation) and}$$
$$\Gamma^{\mathrm{a}}(p) = \operatorname{sgn}(p) = \begin{cases} 1 & \text{for } p \text{ even} \\ -1 & \text{for } p \text{ odd} \end{cases} .$$

sgn(p) is called sign or parity of the permutation p.

Later we will see: There are no other one-dimensional representations of S_n.

Lemma 15. *The symmetriser $s = \sum_{p\in S_n} p$ and the anti-symmetriser $a = \sum_{p\in S_n} \operatorname{sgn}(p)p$ are essentially idempotent and primitive.*

Prove this!

Corresponding irreps: https://youtu.be/syAbXy1vExo (4 min) (2)

Show the non-equivalence of these two irreps using Theorem 13.

Definition: (associate representations)
For a representation Γ^λ of S_n with dimension d_λ, we call Γ^λ and $\widetilde{\Gamma^\lambda} = \Gamma^\lambda \otimes \Gamma^{\mathrm{a}}$ associate representations.

Remarks:

1. Γ^{s} and Γ^{a} are associate to each other.
2. $\dim(\widetilde{\Gamma^\lambda}) = d_\lambda$
3. $\widetilde{\Gamma^\lambda}$ is irreducible $\Leftrightarrow$ Γ^λ is irreducible.
   Why? Recall the irreducibility criterion from Instruction 6.
4. If $\chi^\lambda(p) = 0$ for all odd p, then $\widetilde{\Gamma^\lambda}$ is equivalent to Γ^λ (**why?**), and Γ^λ is called *self-associate*. Otherwise they are non-equivalent.

Theorem 16. *Let Γ^λ and Γ^μ be irreps of S_n. Then*

(i) $\Gamma^\lambda \otimes \Gamma^\mu$ *contains* Γ^{s} *exactly once (not at all),*
if Γ^λ *and* Γ^μ *are equivalent (non-equivalent).*

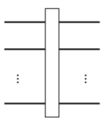
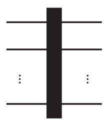
(ii) $\Gamma^\lambda \otimes \Gamma^\mu$ *contains* Γ^{a} *exactly once (not at all),*
if Γ^λ *and* Γ^μ *are associate (not associate).*

Proof:

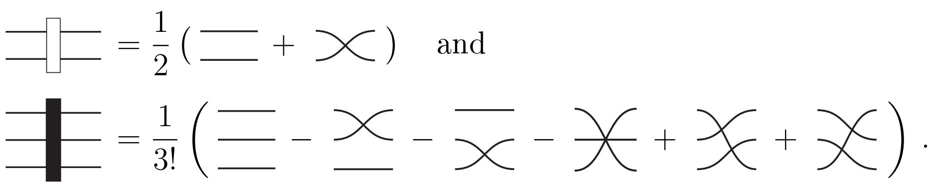
https://youtu.be/OqKaJx422ng (7 min) (3)

5.1.1 Some more birdtracks

In birdtrack notation we denote symmetrisers and anti-symmetrisers by open and solid bars, respectively, i.e.

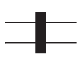
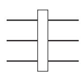
$\frac{1}{n!} s = \frac{1}{n!} \sum_{p \in S_n} p =$  and $\frac{1}{n!} a = \frac{1}{n!} \sum_{p \in S_n} \mathrm{sgn}(p)\, p =$  .

Note that we include a factor of $\frac{1}{n!}$ in the definition of bars over n lines. For instance,

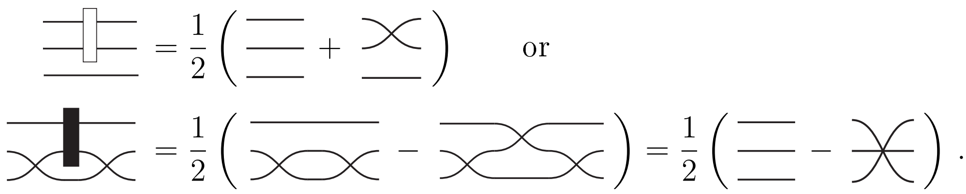


(4)

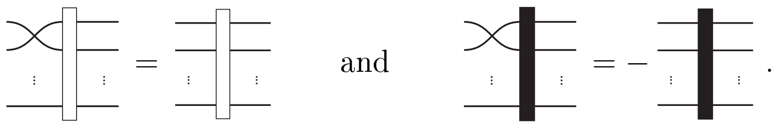
Notice that in birdtrack notation the sign of a permutation, $(-1)^K$, is determined by the number K of line crossings; if more than two lines cross in a point, one should slightly perturb the diagram before counting, e.g.  $\rightsquigarrow$  (K=3).

Expand  and  as in (4).

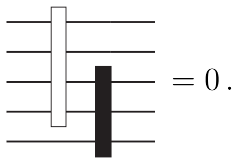
We also use the corresponding notation for partial (anti-)symmetrisation over a subset of lines, e.g.



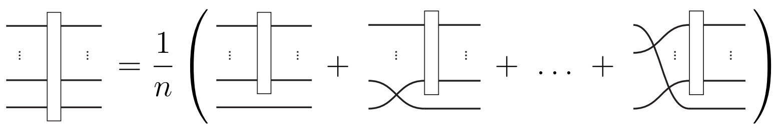
It follows directly from the definition of S and A that when intertwining any two lines S remains invariant and A changes by a factor of (-1), i.e.

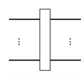


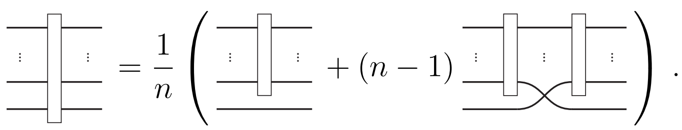
Explain why this implies that whenever two (or more) lines connect a symmetriser to an anti-symmetrizer the whole expression vanishes, e.g.


(5)

Symmetrisers and anti-symmetrisers can be built recursively. To this end notice that on the r.h.s. of



we have sorted the terms according to where the last line is mapped – to the nth, to the $(n-1)$th, ..., to the first line line. Multiplying with  from the left and disentangling lines we obtain the compact relation


(6)

Derive the corresponding recursion relation for anti-symmetrisers.