Universität Tübingen, Mathematisches Institut
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 13 for the preparation of the lecture on 9 June 2021

5.2 Young diagrams and Young tableaux

Definition: (partition, Young diagram)
A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of a natural number n is a (finite) sequence of positive integers with

$$\sum_{i=1}^{r} \lambda_i = n \quad \text{and} \quad \lambda_i \geq \lambda_{i+1} \,.$$

Let λ and μ be two partitions for the same n.

(i) We say that λ and μ are equal, if $\lambda_i = \mu_i \; \forall \, i$.

(ii) We say $\lambda > \mu$ if the first non-vanishing term of the sequence $\lambda_i - \mu_i$ is positive.

Graphically, a partition can be represented by a *Young diagram*:

https://youtu.be/zSsEYZqiCcM (4 min) (1)

Each partition corresponds to a conjugacy class of S_n and vice versa:

https://youtu.be/JMiaRRXWxaU (2 min) (2)

Since the number of Young diagrams with n boxes is equal to the number of conjugacy classes of S_n, it is also equal to the number of non-equivalent irreps of S_n.

Further definitions: *Young tableau*, *normal* Young tableau, and *standard* Young tableau:

https://youtu.be/3oOSaYheyRg (3 min) (3)

The normal Young tableau corresponding to partition λ we denote by Θ_λ. We obtain an arbitrary tableau from Θ_λ by a permutation p of the numbers in the boxes:

$$\Theta_\lambda^p = p\Theta_\lambda \,.$$

This implies $q\Theta_\lambda^p = \Theta_\lambda^{qp}$. **Example:**

$$\Theta_{\boxplus}^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \text{since} \qquad \Theta_{(2,2)} = \Theta_{\boxplus} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Write down all standard tableaux for S_4.

5.3 Young operators

Definitions: Let Θ^p_λ be a Young tableau.
A *horizontal* permutation h^p_λ permutes only numbers within rows of Θ^p_λ.
A *vertical* permutation v^p_λ permutes only numbers within columns of Θ^p_λ.
Furthermore, we define

the *(row-)symmetriser* $$s^p_\lambda = \sum_{\{h^p_\lambda\}} h^p_\lambda\,,$$

the *(column-)anti-symmetriser* $$a^p_\lambda = \sum_{\{v^p_\lambda\}} \mathrm{sgn}(v^p_\lambda)\, v^p_\lambda \quad \text{and}$$

the *Young operator* (or irreducible symmetriser) $$e^p_\lambda = s^p_\lambda\, a^p_\lambda = \sum_{\{h^p_\lambda\}} \sum_{\{v^p_\lambda\}} \mathrm{sgn}(v^p_\lambda)\, h^p_\lambda\, v^p_\lambda\,.$$

Example: standard tableaux for S_3

https://youtu.be/pqO0q2mWiLc (6 min) (4)

Expressed in birdtracks:

https://youtu.be/F19Ol9xUrdE (3 min) (5)

Verify that $e_{\tiny\yng(2,1)}$ is essentially idempotent. Try both, birdtracks and cycle notation.

Observations:

1. For each tableau Θ^p_λ the horizontal and the vertical permutations, $\{h^p_\lambda\}$ and $\{v^p_\lambda\}$, form subgroups of S_n, with $\{h^p_\lambda\} \cap \{v^p_\lambda\} = \{e\}$.
   We obtain the subgroups for Θ^p_λ from those for Θ_λ by conjugation with p, hence $e^p_\lambda = p\, e_\lambda\, p^{-1}$.

2. s^p_λ and a^p_λ are (total) symmetriser and anti-symmetriser of the corresponding subgroup, in the sense that
   $$s^p_\lambda h^p_\lambda = h^p_\lambda s^p_\lambda = s^p_\lambda \qquad \text{and} \qquad a^p_\lambda v^p_\lambda = v^p_\lambda a^p_\lambda = \mathrm{sgn}(v^p_\lambda) a^p_\lambda\,.$$

3. s^p_λ and a^p_λ are essentially idempotent, but in general not primitive.
   The e^p_λ are essentially idempotent and primitive (here for S_3, later for S_n).
   Can you show primitivity for $e_{\tiny\yng(2,1)}$?

4. $e_{\tiny\yng(3)} = s$ and $e_{\tiny\yng(1,1,1)} = a$ generate the two one-dimensional irreps of S_3 (cf. Section 5.1).
   $e_{\tiny\yng(2,1)}$ generates a two-dimensional (minimal) left ideal of $\mathcal{A}(S_3)$:

   https://youtu.be/gX6Q7HzJzSE (6 min) (6)

   $\Rightarrow$ The Young operators of the normal Young tableaux generate all irreps of S_3.

5. **Determine** the (minimal) left ideal generated by $e^{(23)}_{\tiny\yng(2,1)}$.

6. **Verify** that $e = \frac{1}{6} e_{\tiny\yng(3)} + \frac{1}{3} e_{\tiny\yng(2,1)} + \frac{1}{3} e^{(23)}_{\tiny\yng(2,1)} + \frac{1}{6} e_{\tiny\yng(1,1,1)}$ and conclude that the regular rep of S_3 is completely reduced by the Young operators of the standard Young tableaux.