Universität Tübingen, Mathematisches Institut Sommersemester 2021
Dr. Stefan Keppeler

Groups and Representations

Instruction 14 for the preparation of the lecture on 14 June 2021

5.4 Irreducible representations of S_n

Theorem 17. *Let $\lambda \neq \mu$ be partitions of $n \in \mathbb{N}$.*

(i) *The Young operators e^p_λ are essentially idempotent, i.e. $(e^p_\lambda)^2 = \eta_\lambda e^p_\lambda$ with $\eta_\lambda \neq 0$,*

(ii) *the $\frac{1}{\eta_\lambda} e^p_\lambda$ are primitive idempotents.*

(iii) *The irreducible representations generated by e_λ and e_μ are not equivalent.*

(iv) *The irreducible representations generated by e_λ and e^p_λ are equivalent.*

Proof: First notice that no two terms in

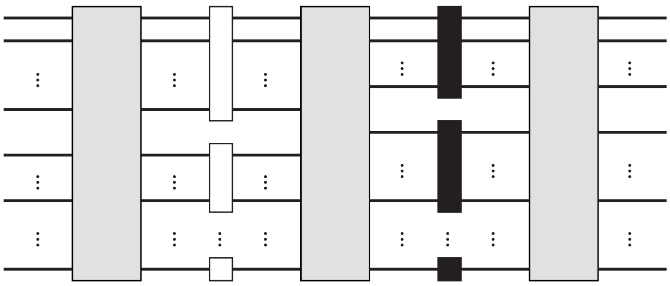
$$e_\lambda = \sum_{\{h_\lambda\}} \sum_{\{v_\lambda\}} \mathrm{sgn}(v_\lambda) h_\lambda v_\lambda$$

are proportional to the same permutation. **Why?** In particular, $e_\lambda \neq 0$ and

$$e_\lambda = e + \text{terms proportional to } p \in S_n \backslash \{e\}\,.$$

In birdtracks we have:

$$e^p_\lambda = \tag{1}$$



https://youtu.be/0e-rjjzijJw (2 min)

With this we can prove all four statements:

(iii)	https://youtu.be/Kq_Z6mnpbXE (7 min)
(i)	https://youtu.be/wEVE7g9w74Y (8 min)
(ii) & (iv)	https://youtu.be/wrh1ILmhthE (4 min)

Remark: Unfortunately, for $n \geq 5$ the Young operators for the standard tableaux no longer satisfy $e_\lambda^p e_\lambda^q = 0 \; \forall \, p \neq q$ (they still satisfy $e_\lambda^p e_\mu^q = 0 \; \forall \, \lambda \neq \mu$, see (iii) above). However, the ideals generated by the Young operators of the standard tableaux are still linearly independent (see exercises) and

$$\mathcal{A}(S_n) = \bigoplus_{\left\{\substack{\text{standard} \\ \text{tableaux}} \Theta_\lambda^p\right\}} \mathcal{A}(S_n) e_\lambda^p \, .$$

(without proof). In particular, this implies that $\dim\left(\mathcal{A}(S_n)e_\lambda^p\right)$ is given by the number of standard tableaux for the partition λ.

5.5 Calculating characters using Young diagrams

The dimension d_λ of irrep Γ^λ is given by the number of standard tableaux for the partition λ. The *hook length formula* (which we won't prove) is very convenient:

$$d_\lambda = \frac{n!}{\prod_{i,j} h_{ij}} \, . \qquad \text{https://youtu.be/DxPI8Q0lh\_Q (3 min)} \tag{2}$$

Determine the dimensions of all irreps of S_4.

Before calculating characters we introduce the notion of a *skew hook:*

https://youtu.be/E_ahyAWIhp0 (2 min) (3)

Here's a recipe (without proof) for calculating characters. Let c be a conjugacy class of S_n with disjoint cycles of lengths $a_1, a_2, \ldots, a_q$. Recursively determine the character χ_c^λ as follows:

- Choose any cycle of c, say with length a_i.
- Denote by $\bar{c}$ the class of S_{n-a_i}, obtained by removing the cycle a_i from c.
- For the Young diagram Θ_λ determine all skew hooks of length a_i and denote the Young diagram(s) of S_{n-a_i}, obtained by removing such a skew hook by $\Theta_{\bar{\lambda}}$. Then

  $$\chi_c^\lambda = \sum_{\bar{\lambda}} \pm \chi_{\bar{c}}^{\bar{\lambda}}$$

  with "+" for positive skew hooks and "−" for negative skew hooks.
- Iterate this procedure.
- If no box of the Young diagram remains then $\chi_{()}^{\bar{\lambda}=0} = 1$.
  (Don't forget the sign of the last skew hook removed!)
- If there is no skew hook of length a_i then $\chi_c^\lambda = 0$.

Example: https://youtu.be/XnSE5E6m6fg (7 min) (4)

Determine the characters of the irrep of S_3 corresponding to .

Explain how we recover the number of standard tableaux when recursively determining the character of the identity.