Universität Tübingen, Mathematisches Institut Sommersemester 2021
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Groups and Representations

Instruction 15 for the preparation of the lecture on 16 June 2021

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6 Lie groups

When speaking about infinite groups we will combine the notion of a group with notions from others areas of mathematics. There will be precise definitions using notions like "topological space", "connectedness" or "differentiable manifold". However, we will not introduce all these notions and concepts in detail. If you are familiar with these notions – fine. If not, don't panic! Some of the subtelties will not be relevant for the cases we are interested in, so we will gloss over them. Aspects which are important in our context will be introduced and discussed carefully, such that no prior knowledge beyond, say, multivariable calculus/analysis in $\mathbb{R}^n$ will be required.

6.1 Topological groups

Definition: (topological group)
A group $(G, \circ)$ is called topological group if

(i) G is a topological space,

(ii) the map $G \ni g \mapsto g^{-1} \in G$ is continuous, and

(iii) the map $G \times G \ni (g, h) \mapsto g \circ h \in G$ is continuous.

Remark: Unless otherwise stated, our topology will be the standard topology on $\mathbb{R}^n$ or $\mathbb{C}^n$, and the induced topology on subspaces.

Examples:

$\mathrm{GL}(n)$, $\mathrm{O}(n)$, $\mathrm{U}(n)$ etc. https://youtu.be/0b_GhknIuNY (3 min) (1)

Definition: (isomomorphism)
Two topological groups G and H are called isomorphic, if there exists a bijective map $f : G \to H$, which is both, an isomorphism of groups, and a homeomorphism of topological spaces (i.e. f is continuous and f^{-1} is continuous).

(Non-)Example:

https://youtu.be/tDCZNBbqYRk (7 min) (2)

Definition: (homogeneous space)
A topological space X is called homogeneous, if for every pair $x, y \in X$ there exists a homeomorphism $f : X \to X$ s.t. $f(x) = y$.

Remark: Every topological group G is homogeneous. This is nice when studying *local* properties.

https://youtu.be/CYu-XpNKu3Q (2 min) (3)

Interesting *global* properties are *compactness* and *connectedness*.

Examples (compactness):

1. $\mathrm{O}(n)$ is compact:

$$\texttt{https://youtu.be/8pFew9Cp-10}\ (3\,\mathrm{min}) \tag{4}$$

2. $\mathrm{O}(1,1)$ is not compact:

$$\texttt{https://youtu.be/zvWRGgcJVpY}\ (2\,\mathrm{min}) \tag{5}$$

3. $\mathrm{GL}(n,\mathbb{R})$ is not compact. **Why?**

Definition: (connected component)
The connected component of $g \in G$ is the union of all connected sets that contain g.

Remarks:

1. A connected component is actually connected.
2. Let $G_0 \subseteq G$ be the connected component of the identity e.
   Show that G_0 is a subgroup of G.
   G_0 is a normal subgroup, and the quotient group G/G_0 is totally disconnected:

$$\texttt{https://youtu.be/-\_mTHYztXUw}\ (4\,\mathrm{min}) \tag{6}$$

Examples:

1. $\mathrm{SU}(2)$ is (simply) connected, since with the parametrisation of Problem 19,

$$\begin{aligned} &\mathrm{SU}(2) \ni \ g = \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix}\,, \\ &|u|^2 + |v|^2 = 1 \quad \Leftrightarrow \quad (\mathrm{Re}\,u)^2 + (\mathrm{Im}\,u)^2 + (\mathrm{Re}\,v)^2 + (\mathrm{Im}\,v)^2 = 1\,, \end{aligned}$$

   $\mathrm{SU}(2)$ is homeomorphic to S^3, and spheres S^n with $n \geq 2$ are (simply) connected.
2. $\mathrm{O}(n)$ is not connected. **Why?**

6.2 Example: SO(2)

Before discussing Lie groups in general, let's look at an example, which illustrates some of the basic ideas. We'll do this in our live session.