Universität Tübingen, Mathematisches Institut
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 16 for the preparation of the lecture on 21 June 2021

6.3 Lie groups

Definition: (Lie group)
A group $(G, \circ)$ is called Lie group if

(i) G is an analytic manifold,
(ii) the map $G \ni g \mapsto g^{-1} \in G$ is analytic, and
(iii) the map $G \times G \ni (g, h) \mapsto g \circ h \in G$ is analytic.

Remarks:

1. An n-dimensional manifold M is something that locally looks like a piece of $\mathbb{R}^n$:

   https://youtu.be/HQMI050AEjw (3 min) (1)

2. Locally, group elements are analytic functions of n parameters:

   https://youtu.be/_VqV9uK76oo (3 min) (2)

3. The so-called *structure constants* $c^j_{k\ell}$ *of the Lie group* are determined by the group law:

   https://youtu.be/4mzmPRyOjgE (6 min) (3)

   Properties of the structure constants:

   (i) For abelian groups $c^j_{k\ell} = 0$, since then $f(x, y) = f(y, x)$.
   (ii) $c^j_{k\ell} = -c^j_{\ell k}$
   (iii) $\sum_\ell (c^j_{k\ell} c^\ell_{nm} + c^j_{n\ell} c^\ell_{mk} + c^j_{m\ell} c^\ell_{kn}) = 0$

   The last identity follows from associativity of group multiplication by comparing the third order terms in the coordinate expansions of $g(h\tilde{g})$ and $(gh)\tilde{g}$.

Examples: matrix Lie groups

1. $\mathrm{GL}(n, \mathbb{R})$ is a Lie group:

   https://youtu.be/so7fTTzjsLo (4 min) (4)

2. For $\mathrm{GL}(n, \mathbb{C})$ consider real and imaginary part of the matrix elements as coordinates and argue as before (in terms of submanifolds of $\mathbb{R}^{2n^2}$).
3. For groups like $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$ or $\mathrm{SU}(n)$ one first observes that they are closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, respectively. One can show that closed subgroups of Lie groups are Lie (sub-)groups. (Later we will study some of these more explicitly.)

6.4 Lie algebras

Definition: A Lie algebra $\mathfrak{g}$ is a vector space over a field K (here mostly $\mathbb{R}$, sometimes $\mathbb{C}$), with an operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
$$(X, Y) \mapsto [X, Y]$$

called *Lie bracket*, which satisfies ($\forall\ X, Y, Z \in \mathfrak{g}$):

(i) $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z] \qquad \forall\ \lambda, \mu \in K$ (linearity)

(ii) $[X, Y] = -[Y, X]$ (anti-symmetry)

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity)

Remarks:

1. A Lie algebra is called commutative if $[X, Y] = 0\ \forall\ X, Y \in \mathfrak{g}$.

2. One can show that the tangent space to a Lie group G at the identity is a Lie algebra $\mathfrak{g}$.

To this end consider curves $g(t)$ in G with $g(0) = e$. Then the derivative (in a chart) at $t = 0$ is a tangent vector.

For matrix Lie groups we can explicitly define the Lie algebra elements, as matrices:

$$-\mathrm{i}\dot{g}(0) := -\mathrm{i}\frac{\mathrm{d}g}{\mathrm{d}t}(0) \in \mathfrak{g}\,.$$

The Lie bracket is now the matrix commutator (rather times $(-\mathrm{i})$, see below)

$$[X, Y] = XY - YX\,.$$

The commutator is linear and anti-symmetric, the Jacobi identity can be verified by direct calculation.

It remains to show that $X, Y \in \mathfrak{g}$ implies that also $(-\mathrm{i})[X, Y] \in \mathfrak{g}$.

https://youtu.be/6VsahKnoDHY (8 min) (5)

3. Choosing a basis $\{X_j\}$ of $\mathfrak{g}$ we have

$$[X_j, X_k] = \mathrm{i}\sum_\ell c^\ell_{jk} X_\ell$$

with the *structure constants* c^ℓ_{jk} *of the Lie algebra* (basis dependent).

The structure constants of the Lie algebra are equal to the structure constants of the corresponding the Lie group (see Section 6.3) – supposing an appropriate choice of basis and coordinates: As basis $\{X_j\}$ for $\mathfrak{g}$ choose the tangent vectors to the coordinate lines in a chart $U \ni e$, i.e. for matrix Lie groups in an explicit parametrisation by taking derivatives with respect to the parameters,

$$X_j = -\mathrm{i}\dot{g}(0) \quad \text{with} \quad g(t) = \varphi^{-1}(0, \ldots, 0, x_j = t, 0, \ldots, 0)\,,$$
$$\text{hence} \quad X_j = -\mathrm{i}\frac{\partial \varphi^{-1}}{\partial x_j}(0)\,.$$

In Section 6.3 we compared expansions of gh and hg, here we essentially expanded $hgh^{-1} - g$. Properties (ii) & (iii) of the structure constants of Section 6.3 now follow from the Lie bracket properties (ii) & (iii) of the commutator.