Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Instruction 18 for the preparation of the lecture on 28 June 2021

6.6 Invariant integration: Haar measure

When representing finite groups we often used the rearrangement lemma as follows

$$\sum_{g\in G} f(g) = \sum_{g\in G} f(hg) = \sum_{g\in G} f(gh) \qquad \forall\, h \in G\,.$$

For continuous groups we would like to replace $\sum_{g\in G} f(g)$ by an integral, say, $\int_G f(g)\,\mathrm{d}\mu(g)$. To this end we need an invariant measure μ.

Theorem 18. (Haar measure)
Every compact topological group possesses a left- and right-invariant measure μ, called Haar measure; it is unique up to normalisation.

(in this generality without proof, but we will explicitly construct μ for compact Lie groups)

Remarks:

1. Invariance means $\quad \mu(gA) = \mu(Ag) = \mu(A) \quad \forall\, g \in G$ and all Borel sets $A \subset G$.
   Shorthand notation: $\mathrm{d}\mu(gh) = \mathrm{d}\mu(hg) = \mathrm{d}\mu(g) \quad \forall\, h \in G$.
   Why does this make sense?
2. For compact groups we will normalise such that $\quad \operatorname{vol} G = \int_G \mathrm{d}\mu(g) = 1\,.$
3. Hence, e.g. for continuous functions,
   $$\int_G f(g)\,\mathrm{d}\mu(g) = \int_G f(hg)\,\mathrm{d}\mu(g) = \int_G f(gh)\,\mathrm{d}\mu(g) \qquad \forall\, h \in G\,. \tag{1}$$
   https://youtu.be/Nx-2sfc_2ro (2 min)
4. Moreover, $\int_G f(g^{-1})\,\mathrm{d}\mu(g) = \int_G f(g)\,\mathrm{d}\mu(g)$.

   https://youtu.be/bRobQky1UQ4 (3 min) (2)
5. Existence implies uniqueness:

   https://youtu.be/keYUKEBcENk (2 min) (3)
6. One also finds invariant measures under weaker conditions. For instance locally compact groups (like $\mathrm{GL}(n,\mathbb{R})$ or the Lorentz group) possess left-invariant and right-invariant measures (unique up to normalisation) but in general the two measures are not identical.

6.6.1 Calculating the Haar measure for a Lie group

Parametrise the group elements using $n = \dim G$ parameters, i.e.[1] $g = g(x_1, \ldots, x_n)$. Then, locally,

$$\mathrm{d}\mu(g) = \varrho(x_1, \ldots, x_n)\, \mathrm{d}^n x$$

with a suitable density $\varrho(x)$ and Lebesgue measure $\mathrm{d}^n x = \mathrm{d}x_1 \ldots \mathrm{d}x_n$.

Hence, under reparametrisation $x = f(y)$ we have:

$$\varrho(x)\, \mathrm{d}^n x = \varrho\big(f(y)\big) \underbrace{\left|\det\left(\frac{\partial f}{\partial y}(y)\right)\right|}_{\text{Jacobian}} \mathrm{d}^n y =: \tilde{\varrho}(y)\, \mathrm{d}^n y$$

We now construct ϱ such that invariance holds.

To this end expand $(-\mathrm{i}) g(x)^{-1} \frac{\partial g}{\partial x_j}(x)$ in a basis $\{X_k\}$ of the Lie algebra $\mathfrak{g}$,

$$g(x)^{-1} \frac{\partial g}{\partial x_j}(x) = \mathrm{i} \sum_k X_k A(x)_{kj}\,. \tag{4}$$

https://youtu.be/-8CiMiXZW0A (4 min)

Claim: The density $\varrho(x) = |\det A(x)|$ defines a left-invariant measure.

Proof:

(i) We first check the behaviour under a (local) change of coordinates $x = f(y)$.

https://youtu.be/0XxGTO9iTF4 (6 min) (5)

(ii) Given a parametrisation in a neighbourhood of g, near $\tilde{g} = hg$ we choose the parametrisation $\tilde{g}(x) = h\, g(x)$. Then $\tilde{\varrho} = \varrho$.

https://youtu.be/kGaGW0p4uvs (5 min) (6)

(iii) Any other parametrisation in a neighbourhood of $\tilde{g}$ can be achieved by a further change of coordinates as in (i). □

What about right-invariance?

Near $\tilde{g} = gh$ choose the parametrisation $\tilde{g}(x) = g(x)h$. Then

$$\mathrm{d}\mu(gh) = |\det \varphi(h)|\, \mathrm{d}\mu(g)\,.$$ https://youtu.be/Waks9edvVJY (6 min) (7)

The factor $|\det \varphi(h)|$ is called *modular function* of G. If $|\det \varphi(h)| = 1\ \forall\, h \in G$, we say that G is unimodular, and the left-invariant measure is also right-invariant.

Compact Lie groups are unimodular.

Show this by studying $\displaystyle\int_G f(gh)\, \mathrm{d}\mu(g)$ for a constant function.

Example: Construct the Haar measure of SO(2) using the parametrisation of Sec. 6.2.

[1] Actually $g = \varphi^{-1}(x_1, \ldots, x_n)$ but here we prefer this shorthand notation.