Universität Tübingen, Fachbereich Mathematik Sommersemester 2021
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Groups and Representations

Instruction 19 for the preparation of the lecture on 30 June 2021

6.7 Properties of compact Lie groups

Theorems 2 and 6 for representations of finite groups also hold for continuous representations of compact Lie groups, if in statements and proofs we replace

$$\frac{1}{|G|}\sum_{g\in G}\dots \qquad \text{by} \qquad \int_G \dots \, \mathrm{d}\mu(g)\,,$$

Hence:

(i) Every finite-dimensional representation is equivalent to a unitary representation.

(ii) Matrix elements of unitary irreps Γ^μ, Γ^ν (non-equivalent for $\mu \neq \nu$) are orthogonal,

$$\int_G \overline{\Gamma^\mu(g)_{jk}}\, \Gamma^\nu(g)_{j'k'} \, \mathrm{d}\mu(g) = \frac{1}{d_\mu}\delta_{\mu\nu}\delta_{jj'}\delta_{kk'}\,,$$

with $d_\mu = \dim \Gamma^\mu$.

(iii) Similarly for the characters $\chi^\mu(g) = \operatorname{tr}\Gamma^\mu(g) = \sum_j \Gamma^\mu(g)_{jj}$,

$$\int_G \overline{\chi^\mu(g)}\, \chi^\nu(g)\, \mathrm{d}\mu(g) = \delta_{\mu\nu}\,.$$

This implies again

$$\Gamma \text{ is irreducible} \quad \Leftrightarrow \quad \int_G |\chi(g)|^2 \, \mathrm{d}\mu(g) = 1\,,$$

as well as: If Γ is a directe sum of irreps, $\Gamma = \bigoplus_\mu a_\mu \Gamma^\mu$, then

$$a_\mu = \int_G \overline{\chi^\mu(g)}\, \chi(g)\, \mathrm{d}\mu(g)\,.$$

For finite groups we also showed completeness of the representation matrices' elements (Problem 16) and complete reducibility of the regular representation, carried by the group algebra $\mathcal{A}(G)$ (Section 2.7). This implied that there were only finitely many non-equivalent irreps.

Similarly one can show that compact Lie groups have countably many non-equivalent (continuous) irreducible representations, which are all of finite dimension. Moreover, every continuous representation is a direct sum of irreducible representations. All this follows from the *Peter-Weyl theorem*.

Consider the vector $L^2(G)$ of functions $\phi : G \to \mathbb{C}$, with scalar product

$$\langle \phi | \psi \rangle = \int_G \overline{\phi(g)}\, \psi(g)\, \mathrm{d}\mu(g)\,.$$

The role of the regular representation is played by Γ defined as

$$\big(\Gamma(h)\phi\big)(g) = \phi(h^{-1}g) \qquad \forall\ h \in G\,.$$

Convince yourself that Γ is a representation.

Does it make sense that functions $\phi : G \to \mathbb{C}$ now play the role that elements of $\mathcal{A}(G)$ played for finite groups?

Theorem 19. (Peter-Weyl)
Let G be a compact Lie group with non-equivalent irreps Γ^μ, $\dim \Gamma^\mu = d_\mu$. Then the matrix elements $\sqrt{d_\mu}\, \Gamma^\mu(g)_{jk}$, $j, k = 1, \ldots, d_\mu$, form a complete set of orthonormal functions for $L^2(G)$.

(without proof)

Remarks:

1. We can thus expand every function $\phi \in L^2(G)$ as

$$\phi(g) = \sum_{\mu,j,k} c_{\mu jk}\, \Gamma^\mu(g)_{jk} \qquad \text{with} \qquad c_{\mu jk} = d_\mu \int_G \overline{\Gamma^\mu(g)_{jk}}\, \phi(g)\, \mathrm{d}\mu(g)$$

   (convergence in L^2-sense).

   What does this reduce to for $G = \mathrm{SO}(2) \cong \mathrm{U}(1)$? (cf. Section 6.2)

2. In physics notation we write completeness as

$$\sum_{\mu,j,k} d_\mu\, \Gamma^\mu(g)_{jk}\, \overline{\Gamma^\mu(h)_{jk}} = \delta(g-h) \qquad \text{with} \qquad \int_G \delta(g-h)\, f(g)\, \mathrm{d}\mu(g) = f(h)\,.$$

6.8 Irreducible representations of SO(3)

For every $g \in \mathrm{SO}(3)$ exists an $X \in \mathfrak{so}(3)$ s.t. $g = \mathrm{e}^{\mathrm{i}X}$. Choose, e.g., the basis

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

of $\mathfrak{so}(3)$ with

$$[J_j, J_k] = \mathrm{i} \sum_\ell \varepsilon_{jk\ell} J_\ell \,.$$

Then

$$R_{\vec{n}}(\psi) = \mathrm{e}^{-\mathrm{i}\psi \vec{n}\vec{J}} \qquad \text{where} \qquad \vec{n}\vec{J} = \sum_{j=1}^{3} n_j J_j$$

(rotation about axis $\vec{n}$ by angle ψ, cf. Section 6.5):

https://youtu.be/uPs-QSVt13s (6 min) (1)

From every representation of a Lie group we obtain (by taking derivatives) a representation of the corresponding Lie algebra (in terms of matrices). More precisely, with $g(t)$, $g(0) = e$, $\dot{g}(0) = \mathrm{i}X$ and a rep Γ of G define the derived rep $\mathrm{d}\Gamma$ of $\mathfrak{g}$ by

$$\mathrm{d}\Gamma(X) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma\big(g(t)\big)\Big|_{t=0} \,.$$

From a representation of the Lie algebra $\mathfrak{so}(3)$ we obtain (by exponentiating) a representation of the group SO(3), if the global (topological) properties match those of SO(3).

Construction of reps of $\mathfrak{so}(3)$. The matrix (operator)

$$J^2 = \sum_{j=1}^{3} J_j^2$$

commutes with every $X \in \mathfrak{so}(3)$:

https://youtu.be/sxbtMVW2PJA (5 min) (2)

Remark: J^2 is not in the Lie algebra; it is a so-called Casimir operator and an element of the enveloping algebra (see later).

Consequences:

1. $[J^2, X] = 0 \;\forall\, X \in \mathfrak{so}(3)$ implies $[J^2, g] = 0 \;\forall\, g \in \mathrm{SO}(3)$. **Why?**
2. Consider a representation of SO(3). Now all this also holds for the representation matrices of g, X, and J^2.
3. If the representation is irreducible then according to Schur's Lemma (Theorem 4), the representation matrix of J^2 is a multiple of the identity matrix.

Next time we will construct all irreps of $\mathfrak{so}(3)$ in terms of simultaneous eigenvectors of the representation matrices of J^2 and one generator. After exponentiation these become irreps of SO(3) if the global properties are correct.