Universität Tübingen, Fachbereich Mathematik Sommersemester 2021
Dr. Stefan Keppeler

Groups and Representations

Instruction 20 for the preparation of the lecture on 5 July 2021

6.8 Irreducible representations of SO(3) (cont.)

Assume we are given a representation of SO(3).

Notation: We denote the representation matrices of g, X, J^2 also by g, X, J^2.

Construct irreducible subspaces (and thus irreps) as follows:

- Choose a suitable starting vector.
- Generate an irreducible basis by repeatedly applying the generators.

Suitable starting vector: Joint normalised eigenvector of J^2 and J_3 (**Why** can we choose it in this way?), in Dirac notation

$$J_3|m\rangle = m|m\rangle$$

We define $J_\pm = J_1 \pm \mathrm{i}J_2$. Then

$$[J_\pm, J_3] = \mp J_\pm \qquad \text{and thus} \qquad J_3 J_\pm|m\rangle = (m \pm 1)J_\pm|m\rangle\,, \tag{1}$$

https://youtu.be/4RE3ZFSPyGI (4 min)

i.e. either $J_\pm|m\rangle \propto |m \pm 1\rangle$ or $J_\pm|m\rangle = 0$.

Since the invariant subspace has to be finite dimensional the sequence

$$\ldots, J_-|m\rangle, |m\rangle, J_+|m\rangle, J_+^2|m\rangle, \ldots$$

has to terminate on both sides, say at $m = j$ and at $m = \ell$ with $j \geq \ell$,

$$J_3|j\rangle = j|j\rangle\,, \qquad J_3|\ell\rangle = \ell|\ell\rangle\,,$$
$$J_+|j\rangle = 0\,, \qquad J_-|\ell\rangle = 0\,.$$

What is the dimension of this irreducible subspace?

We further have

$$J^2 = J_3^2 + J_-J_+ + J_3 \qquad \text{and} \qquad J^2 = J_3^2 + J_+J_- - J_3\,, \tag{2}$$

https://youtu.be/-qlcOB1JBmo (2 min)

and in particular (**why?**)

$$J^2|j\rangle = (J_3^2 + J_3 + J_-J_+)|j\rangle = j(j+1)|j\rangle\,,$$
$$J^2|\ell\rangle = (J_3^2 - J_3 + J_+J_-)|\ell\rangle = \ell(\ell-1)|\ell\rangle\,.$$

Since both eigenvalues have to be identical (**why?**) we conclude that $\ell = -j$ (**why?**) and $j \geq 0$. Hence, we can label $\mathfrak{so}(3)$ irreps by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

- The dimension of irrep j is $2j + 1$.

From now on denote orthonormal basis vectors as $|jm\rangle$. Then

$$\begin{aligned} J^2|jm\rangle &= j(j+1)|jm\rangle \\ J_3|jm\rangle &= m|jm\rangle \\ J_\pm|jm\rangle &= [j(j+1) - m(m\pm 1)]^{1/2}|j, m\pm 1\rangle \end{aligned}$$

Verify the last identity by calculating the norm of $J_\pm|jm\rangle$.

Irreps of SO(3). Now we distinguish again between g and $\Gamma(g)$ and between X and $\mathrm{d}\Gamma(X)$. Denote by Γ^j the *potential* irrep of SO(3) carried by $\{|jm\rangle : m = -j \ldots, j\}$, i.e. the matrix elements are

$$\Gamma^j\left(\mathrm{e}^{-\mathrm{i}\psi\vec{n}\vec{J}}\right)_{mm'} = \langle jm|\,\mathrm{e}^{-\mathrm{i}\psi\,\mathrm{d}\Gamma(\vec{n}\vec{J})}\,|jm'\rangle\,.$$

Only for integer j does this define a representation of SO(3):

https://youtu.be/Cviw6oLYN68 (5 min) (3)

Irreps of SU(2). The Pauli matrices σ_1, σ_2, σ_3 (see Problem 32) form a basis of the Lie algebra $\mathfrak{su}(2)$ with

$$[\sigma_j, \sigma_k] = 2\mathrm{i}\sum_l \varepsilon_{jkl}\sigma_l\,,$$

i.e. the matrices $\sigma_k/2$ satisfy the same relations as the J_k, and thus $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Hence we also already know all irreps of $\mathfrak{su}(2)$. Since $\mathrm{SU}(2) = \exp(\mathrm{i}\mathfrak{su}(2))$ (cf. Problems 32 & 34), we get irreps of SU(2) for all $j \in \mathbb{N}_0/2$.

Determine the characters of all irreps of SO(3) and of all irreps of SU(2).

6.9 Remarks on some classical Lie groups

Definition: (adjoint representation)
Let G be a (matrix) Lie group with corresponding Lie algebra $\mathfrak{g}$, and let $g \in G$. The map $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g$ with

$$\begin{aligned} \mathrm{Ad}_g : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto gXg^{-1} = \mathrm{Ad}_g(X) \end{aligned}$$

is called adjoint representation of G (on $\mathfrak{g}$).

Remarks:

1. Ad is actually a representation. **Show this.**
2. We also define $\mathrm{Ad}_g(h) = ghg^{-1}$ for $h \in G$.
3. For $X \in \mathfrak{g}$ we further define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}_X(Y) = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{\mathrm{e}^{\mathrm{i}Xt}}(Y) \Big|_{t=0} = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{\mathrm{i}Xt} Y \mathrm{e}^{-\mathrm{i}Xt} \right) \Big|_{t=0} = [X, Y] \, .$$

Lemma 20. (Principal axis theorem for unitary matrices)
For every $g \in \mathrm{U}(n)$ there exists an $h \in \mathrm{U}(n)$ s.t. $h^\dagger g h$ is diagonal, in particular

$$g = h \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{\mathrm{i}\varphi_n} \end{pmatrix} h^\dagger$$

with real φ_j.

Proof: Reduce to the principal axis theorem for Hermitian matrices.
Let $M_\phi := \{ g \in \mathrm{U}(n) \, : \, \mathrm{e}^{\mathrm{i}\phi} \text{ is not eigenvalue of } g \}$. Then

$$\begin{aligned} f_\phi : M_\phi &\to \mathbb{C}^{n \times n} \\ g &\mapsto \mathrm{i}(\mathrm{e}^{\mathrm{i}\phi} + g)(\mathrm{e}^{\mathrm{i}\phi} - g)^{-1} \end{aligned}$$

(generalised *Cayley transformation*) maps unitary g to Hermitian matrices $A = f_\phi(g)$:

https://youtu.be/MDeXGKn0odo (3 min) (4)

Now there exists an $h \in \mathrm{U}(n)$ s.t. $h^\dagger A h = D$ is diagonal (principal axis theorem for Hermitian matrices). Furthermore, we can explicitly invert f_ϕ:

https://youtu.be/BZg9dikdumE (2 min) (5)

Finally, for given $g \in \mathrm{U}(n)$ choose ϕ s.t. $g \in M_\phi$, call $A = f_\phi(g)$, and choose $h \in \mathrm{U}(n)$ s.t. $h^\dagger A h = D$ is diagonal. Then h also diagonalises g:

$$h^\dagger g h = h^\dagger \mathrm{e}^{\mathrm{i}\phi} (A + \mathrm{i})^{-1} h h^\dagger (A - \mathrm{i}) h = \mathrm{e}^{\mathrm{i}\phi} (D + \mathrm{i})^{-1} (D - \mathrm{i}) \, .$$

□

Explain why the analogous result also holds for $g \in \mathrm{SU}(n) \subset \mathrm{U}(n)$, with $h \in \mathrm{SU}(n)$.