Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

# Groups and Representations

Instruction 21 for the preparation of the lecture on 7 July 2021

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## 6.9 Remarks on some classical Lie groups (cont.)

**Theorem 21.** *For every $g \in \mathrm{U}(n)$ there exists an $X \in \mathfrak{u}(n)$ s.t. $g = \mathrm{e}^{\mathrm{i}X}$.*

**Proof:**

https://youtu.be/2SganmGZf7k (2 min) (1)

**Remarks:**

1. Simillarly, for every $g \in \mathrm{SU}(n)$ there exists an $X \in \mathfrak{su}(n)$, s.t. $g = \mathrm{e}^{\mathrm{i}X}$. **Why?**
2. Similarly for $g \in \mathrm{SO}(2n)$: One first shows that there exists an $h \in \mathrm{SO}(2n)$ s.t.

$$g = h \begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_n \end{pmatrix} h^T$$

   with $R_j \in \mathrm{SO}(2)$. **Why?** For $\mathrm{SO}(2n+1)$ the diagonal matrix has an additional row with an entry 1. Then also every $g \in \mathrm{SO}(n)$ can be written as $\mathrm{e}^{\mathrm{i}X}$ with $X \in \mathfrak{so}(n)$.
3. In all these cases we can in principle construct irreps using the same strategy as in Section 6.8 for SO(3) or SU(2): First construct irreducible representations of the Lie algebra and by exponentiation (potential) reps of the group.
4. The diagonal matrices which appear in this procedure are maximal abelian subgroups (so-called *maximal tori*) of the corresponding group.

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## 6.10 More on Lie algebras and related topics

**Definition:** (representation of a Lie algebra)
Let $\mathfrak{g}$ be a Lie algebra and $V$ a vector space. A representation $\phi$ is a linear map that assigns to each $X \in \mathfrak{g}$ a linear map $\phi(X) : V \to V$ s.t.

$$\phi(\mathrm{i}\underbrace{[X,Y]}_{\text{Lie bracket}}) = \underbrace{[\phi(X), \phi(Y)]}_{\text{commutator}} \qquad \forall\, X, Y \in \mathfrak{g}\,.$$

**Remark:** The i-decoration comes from our convention that $G = \exp(\mathrm{i}\mathfrak{g})$.

**Examples:**

1. $\mathrm{ad} : \mathfrak{g} \ni X \mapsto \mathrm{ad}_X$ with $\mathrm{ad}_X(Y) = [X, Y]$ defines a representation of $\mathfrak{g}$ on $\mathfrak{g}$:

https://youtu.be/wfkch23mM04 (5 min) (2)

2. From a rep $\Gamma$ of a Lie group $G$ we obtain (by differentiation) a rep $\mathrm{d}\Gamma$ of the Lie algebra $\mathfrak{g}$,

$$\mathrm{d}\Gamma(X) = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} \Gamma(\mathrm{e}^{\mathrm{i}Xt})\Big|_{t=0} .$$

**Definition:** (enveloping algebra)
Let $\mathfrak{g}$ be a Lie algebra with basis $\{X_j\}$. The enveloping algebra $E(\mathfrak{g})$ consists of formal polynomials in the generators

$$\sum_j a_j(\mathrm{i}X_j) + \sum_{jk} b_{jk}(\mathrm{i}X_j)(\mathrm{i}X_k) + \sum_{jkl} c_{jkl}(\mathrm{i}X_j)(\mathrm{i}X_k)(\mathrm{i}X_l) + \dots , \qquad a_j, b_{jk}, c_{jkl} \in \mathbb{R} ,$$

where $\mathrm{i}X_j\mathrm{i}X_k$ and $\mathrm{i}X_k\mathrm{i}X_j + \mathrm{i}X_l$ have to be identified if $[\mathrm{i}X_j, \mathrm{i}X_k] = \mathrm{i}X_l$.

**Remarks:**

1. A representation $\phi$ of a Lie algebra then also induces a representation of the enveloping algebra, whereby the formal products and sums become matrix products and matrix sums.
2. A basis of the enveloping algebra is, e.g., given by those monomials in the generators for which the indices are non-decreasing from left to right:

https://youtu.be/tydgmSEW18I (2 min) (3)

**Definition:** (Casimir operator)
$C \in E(\mathfrak{g})$ is called Casimir operator if $C$ commutes with all elements of the enveloping algebra, i.e. if

$$[C, A] = 0 \quad \forall\ A \in E(\mathfrak{g}) .$$

**Example:** $J^2 := J_1^2 + J_2^2 + J_3^2$ for SO(3) (cf. Section 6.8).

**Remarks:**

1. In particular a Casimir operator commutes with all $X \in \mathfrak{g} \subseteq E(\mathfrak{g})$.
2. This implies $\mathrm{e}^{\mathrm{i}X} C \mathrm{e}^{-\mathrm{i}X} = C\ \forall\ X \in \mathfrak{g}$, i.e. in the cases of Sections 6.8 and 6.9, where $G = \exp(\mathrm{i}\mathfrak{g})$, we immediately conclude $gCg^{-1} = C\ \forall\ g \in G$.
3. $gCg^{-1} = C\ \forall\ g \in G$ is even true more generally, since one can show:
    - $\exp(\mathrm{i}\mathfrak{g})$ always contains a neighbourhood of the identity in $G$.
    - By taking (finite) products $\mathrm{e}^{\mathrm{i}X}\mathrm{e}^{\mathrm{i}Y}\mathrm{e}^{\mathrm{i}Z}\dots$ one reaches all $g \in G_0$, the connected component of the identity.
    - If $G$ is connected, then for representations (of the Lie group, the Lie algebra and the enveloping algebra) we thus have $[\mathrm{d}\Gamma(C), \Gamma(g)] = 0\ \forall\ g \in G$, and according to Schur's Lemma (Theorem **??**) it follows that for irreps $\mathrm{d}\Gamma(C)$ is a scalar multiple of $\mathbb{1}$.