Universität Tübingen, Fachbereich Mathematik Sommersemester 2021
Dr. Stefan Keppeler

Groups and Representations

Instruction 22 for the preparation of the lecture on 12 July 2021

---

7 Tensor method for constructing irreps of GL(N) and subgroups

7.1 Setting

In the following let V be complex vector space with $\dim V = N$, i.e. $V \cong \mathbb{C}^N$.

Define $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ factors}}$. Form tensor products from $|v_j\rangle \in V$, $j = 1, \dots, n$:

$$\bigotimes_{j=1}^{n} |v_j\rangle = |v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle \in V^{\otimes n} .$$

General $|v\rangle \in V^{\otimes n}$ are linear combinations of tensor products, called *tensors of rank n*.

$V^{\otimes n}$ carries reps Γ of GL(N) and D of S_n

$$\Gamma(g) \bigotimes_{j=1}^{n} |v_j\rangle = \bigotimes_{j=1}^{n} \gamma(g)|v_j\rangle , \qquad \text{with } \gamma(g) = g \text{ (defining rep)},$$

$$D(p)\big(|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle\big) = |v_{p^{-1}(1)}\rangle \otimes |v_{p^{-1}(2)}\rangle \otimes \cdots \otimes |v_{p^{-1}(n)}\rangle ,$$

everything continued by linearity; D also extends to a rep of $\mathcal{A}(S_n)$.

Convince yourself that D is a rep.

These reps commute, i.e.

$$\Gamma(g)D(p)|v\rangle = D(p)\Gamma(g)|v\rangle \qquad \forall\, p \in S_n\,,\ \forall\, g \in \mathrm{GL}(N)\,,\ \forall\, |v\rangle \in V^{\otimes n} ,$$

and even $\forall\, p \in \mathcal{A}(S_n)$.

Notation: Form now on, we omit Γ and D, e.g. we write $gp|v\rangle = pg|v\rangle$.

How does $p \in S_n$ act on an arbitrary $|x\rangle \in V^{\otimes n}$?

https://youtu.be/EEEq-bCuc5c (3 min) (1)

7.2 Decomposition of $V^{\otimes n}$ into irreducible invariant subspaces with respect to S_n and GL(N)

7.2.1 Symmetry classes

Let Θ_λ^p be a Young tableau, e_λ^p the corresponding Young operator, and $L_\lambda = \mathcal{A}(S_n)e_\lambda$ the minimal left ideal generated by e_λ

In the following we will see:

For fixed $\|v\rangle \in V^{\otimes n}$ the subspace $L_\lambda\|v\rangle = \mathcal{A}(S_n)e_\lambda\|v\rangle$ (if non-empty) is invariant and irreducible w.r.t. S_n.	The subspace $e_\lambda^p V^{\otimes n}$ is invariant and irreducible w.r.t. GL(N).

Then we will be able to choose a basis $\{|\lambda,\alpha,a\rangle\}$ of $V^{\otimes n}$ s.t.
λ lables the so-called symmetry class, given by a Young diagram,
α labels the irreducible invariant subspaces w.r.t. S_n,
a labels the irreducible invariant subspaces w.r.t. GL(N).

Lemma 22. *For fixed $|\alpha\rangle \in V^{\otimes n}$ the subspace $T_\lambda(\alpha) = L_\lambda|\alpha\rangle$ is either empty or*

(i) *$T_\lambda(\alpha)$ is invariant and irreducible under S_n and*

(ii) *the S_n irrep carried by $T_\lambda(\alpha)$ is given by the irrep carried by L_λ.*

Proof:

https://youtu.be/Nv1AecrF2vE (6 min) (2)

7.2.2 Totally symmetric and totally anti-symmetric tensors

Let $\lambda = \mathrm{s} = \square\square\cdots\square$, i.e. $e_\mathrm{s} = s$ is the total symmetriser of S_n, L_s is one-dimensional.

$\Rightarrow$ For given $|\alpha\rangle$ the subspace $T_\mathrm{s}(\alpha)$ is one-dimensional, $T_\mathrm{s}(\alpha) = \mathrm{span}(e_s|\alpha\rangle)$.

These tensors are *totally symmmetric* (in all indices).

Each $T_\mathrm{s}(\alpha)$ carries the trivial representation of S_n.

Example: $n = 3,\ N = 2$

https://youtu.be/Crhbo74Jj0k (5 min) (3)

We denote the space spanned by the tensors of symmetry class s by T'_s.

Totally anti-symmetric tensors exist only for $n \leq N$,

$\lambda = \mathrm{a} = \begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline\end{array}\,,$ since for $n > N$ every basis vector contains at least two identical indices, anti-symmetrisation yields zero.

The S_n irrep carried by $T_\mathrm{a}(\alpha)$ is sgn.

Example: $n = 2,\ N \geq 2$

https://youtu.be/sX_vkzbmiiQ (2 min) (4)

Construct all totally symmetric tensors for $n = 2$ and arbitrary N. How many are there?