Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 23 for the preparation of the lecture on 14 July 2021

7.2.3 Tensors with mixed symmetry

Example: Consider tensors of rank $n = 3$ in $N = 2$ dimensions, and choose

$$\Theta_\kappa = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \qquad \text{with} \qquad e_\kappa = e + (12) - (13) - (132)\,.$$

From Section 5.3 we know: $L_\kappa = \operatorname{span}(e_\kappa, (23)e_\kappa)$

▶ First we choose $|\alpha\rangle = |112\rangle$:

$$e_\kappa|112\rangle = 2|112\rangle - |211\rangle - |121\rangle \qquad =: |\kappa, 1, 1\rangle\,,$$
$$(23)e_\kappa|112\rangle = 2|121\rangle - |211\rangle - |112\rangle \qquad =: |\kappa, 1, 2\rangle\,.$$

https://youtu.be/saVR889k6qA (3 min)

$\Rightarrow T_\kappa(1) = \mathcal{A}(S_3)e_\kappa|112\rangle = \operatorname{span}(|\kappa,1,1\rangle, |\kappa,1,2\rangle)$ is invariant and irreducible under S_3.

▶ Then we choose $|\alpha\rangle = |221\rangle$:

$$e_\kappa|221\rangle = 2|221\rangle - |122\rangle - |212\rangle] =: |\kappa, 2, 1\rangle\,,$$
$$(23)e_\kappa|221\rangle = 2|212\rangle - |122\rangle - |221\rangle] =: |\kappa, 2, 2\rangle\,,$$

$\Rightarrow T_\kappa(2) = \mathcal{A}(S_3)e_\kappa|221\rangle = \operatorname{span}(|\kappa,2,1\rangle, |\kappa,2,2\rangle)$ is invariant and irreducible under S_3.

▶ $|\kappa,1,1\rangle$ and $|\kappa,2,1\rangle$ span the 2-dimensional subspace $T'_\kappa(1) = e_\kappa V^{\otimes 3}$.

(i) $T'_\kappa(1)$ is invariant under GL(2), since $gp = pg\ \forall\, g \in \mathrm{GL}(2)$ and $\forall\, p \in S_3$ implies

$$g e_\kappa|v\rangle = e_\kappa g|v\rangle \in T'_\kappa(1)\,.$$

This argument requires neither $n = 3$ nor $N = 2$, nor $\lambda = \kappa$ – it is true in general!

(ii) $T'_\kappa(1)$ is irreducible under GL(2).

Proof: We explicitly construct the representation matrices for $g \in \mathrm{GL}(2)$:

$$\Gamma^\kappa(g) = \det g \begin{pmatrix} g_{11} & -g_{12} \\ -g_{21} & g_{22} \end{pmatrix} \qquad (1)$$

https://youtu.be/fN3r9Ja6wkU (10 min)

Why does this prove irreducibility of Γ^κ?

▶ Similarly, $|\kappa,1,2\rangle$ and $|\kappa,2,2\rangle$ span $T'_\kappa(2) = e_\kappa^{(23)}V^{\otimes 3}$, which is also invariant and irreducible under GL(2) and carries a rep that is equivalent to Γ^κ.

▶ The direct sum $T'_\kappa(1) \oplus T'_\kappa(2)$ contains all tensors of symmetry class $\kappa = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \\ \cline{1-1}\end{array}$.

Summary: Complete reduction of the 8-dimensional space $V^{\otimes 3}$:
(recall that $\Theta_{\mathrm{s}} = \yng(3)$ and $\Theta_\kappa = \yng(2,1)$)

$$\begin{aligned} V^{\otimes 3} &= \underbrace{T_{\mathrm{s}}(1) \oplus T_{\mathrm{s}}(2) \oplus T_{\mathrm{s}}(3) \oplus T_{\mathrm{s}}(4)}_{} \;\; \oplus \underbrace{T_\kappa(1) \oplus T_\kappa(2)}_{} && \leftarrow \text{invariant under } S_3 \\ &= \qquad\qquad T'_{\mathrm{s}} \qquad\qquad \oplus \overbrace{T'_\kappa(1) \oplus T'_\kappa(2)}^{} && \leftarrow \text{invariant under } \mathrm{GL}(2) \end{aligned}$$

T'_{s} carries a 4-dimensional irrep of GL(2); under S_3 it is the direct sum of 4 one-dimensional subspaces, each carrying the trivial rep.

As a convenient basis for $V^{\otimes 3}$ we can choose:

- the 4 totally symmetric tensors from Section 7.2.2, and
- the 4 tensors $|\kappa, \alpha, a\rangle$ with $\alpha = 1, 2$ and $a = 1, 2$.

7.2.4 Complete reduction of $V^{\otimes n}$

The observations and results of the preceding sections generalise. We will look at this together in our live session.

7.2.5 Dimensions of the GL(N) irreps

Essentially, we already know the dimensions of the GL(N) irreps: To each Young diagram λ corresponds an S_n irrep D^λ and a GL(N) irrep Γ^λ. For the S_n irreps we can determine dimensions and multiplicities (within $V^{\otimes n}$) using the methods of Sections 4.3.1 and 5.5. According to the construction in Sections 7.2.1–4 the multiplicity of D^λ is equal to the dimension of Γ^λ and vice versa. Determining the dimensions in this way can be tedious, and there are several other algorithms and formulae. We will speak about the following two in our live session:

- **Graphical rule:** The dimension of the GL(N) irrep corresponding to the Young diagram λ is given by the number of *semi-standard* Young tableaux Θ_λ. In semi-standard Young tableaux numbers need not increase in rows but must only be non-deceasing.

  Example: For $N = 2$ we find

  $$\dim \Gamma^{\yng(2,1)} = 2 \qquad \text{and} \qquad \dim \Gamma^{\yng(3)} = 4\,,$$

  since the allowed choices are

  $$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}\,,\; \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \qquad \text{and} \qquad \begin{ytableau} 1 & 1 & 1 \end{ytableau}\,,\; \begin{ytableau} 1 & 1 & 2 \end{ytableau}\,,\; \begin{ytableau} 1 & 2 & 2 \end{ytableau}\,,\; \begin{ytableau} 2 & 2 & 2 \end{ytableau}\,.$$

  Determine the corresponding dimensions for $N = 3$.

- **Hook length formula:**

  $$\dim(\Gamma^\lambda) = \prod_{ij} \frac{N + j - i}{\underset{\substack{\uparrow \\ \text{hook length of box } i,j}}{h_{ij}}} \qquad \begin{pmatrix} \text{product over all boxes of } \lambda \\ i = \text{row},\; j = \text{column index} \end{pmatrix}$$