Universität Tübingen, Fachbereich Mathematik
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Sommersemester 2021

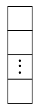
Groups and Representations

Instruction 24 for the preparation of the lecture on 19 July 2021

7.3 Irreps of U(N) and SU(N)

The GL(N) irreps from Section 7.2 restrict to representations of subgroups, which do not need to be irreducible. They are, however, irreducible for U(N) and SU(N) but in general not for O(N) and SO(N).

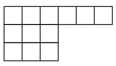
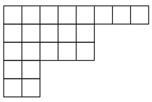
U(N) and SU(N)	https://youtu.be/WM6vX88PKG4 (4 min)	(1)
O(N) and SO(N)	https://youtu.be/_-ooGDPg204 (4 min)	(2)

Show that the GL(N) irrep corresponding to the Young diagram **a** =  with N rows is given by the determinant:

- ▶ First recall that for vectors $|i_1, \ldots, i_N\rangle$ contributing to $e_{\mathrm{a}} g|\alpha\rangle$ all i_k are different.
- ▶ Write these vectors as $p|1, \ldots, N\rangle$ with a permutation p.
- ▶ Then calculate $e_{\mathrm{a}} g|1, \ldots, N\rangle$ for $g \in \mathrm{GL}(N)$.

Which irrep corresponds to **a** if we replace GL(N) by the subgroup SU(N)?

In the exercises we will show that the SU(N) irreps corresponding to the Young diagrams (with row lenghts) $(\lambda_1, \ldots, \lambda_N)$ and $(\lambda_1+k, \ldots, \lambda_N+k)$ are equivalent, e.g.

 and  for $N = 5$ and $k = 2$.

For SU(2), except for the trivial rep, all irreps can be labelled by one-row Young diagrams. **What** are the corresponding dimensions?

7.4 Reducing tensor products in terms of Young diagrams

Goal: Given two irreps Γ^λ and $\Gamma^{\lambda'}$ of GL(N), U(N) or SU(N) with Young diagrams λ and λ' find the complete reduction of the product rep $\Gamma^\lambda \otimes \Gamma^{\lambda'}$.

Examples and observations:

$\square^{\otimes 2}$, $\square^{\otimes 3}$, $\square^{\otimes 4}$	https://youtu.be/FLPJbunjr9U (11 min)	(3)

Closer inspections leads to the Littlewood-Richardson rule (which we won't prove):

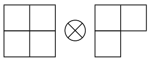
1. Write the number i in all boxes of row i of λ'.
2. Add the boxes of λ' to λ, first the 1s, then the 2s etc. adhering to the following rules:

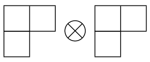
(a) In each step the resulting diagram has to be a valid Young diagram and must not have more than N rows.
(b) No number may appear more than once in the same column.
(c) When reading the numbers row-wise from *right to left* beginning with the first row, then the second etc., and terminating this sequence at any point, there must never be more is than $(i{-}1)$s.

3. For SU(N) columns with N boxes can be omitted.

Always check your result by comparing dimensions on both sides of the equation.

Example:

 for SU(3) https://youtu.be/xrze6-yRWTI (10 min) (4)

Reduce  for SU(3).

7.5 Complex conjugate representations

Observation: Sometimes $\dim\Gamma^{\lambda} = \dim\Gamma^{\lambda'}$ for $\lambda \neq \lambda'$. This may be "accidental" but often it can be understood systematically in terms of the following construction.

Example: Consider  for $N = 3$.

Basis tensors: (anti-symmetric tensors of rank 2 in 3 dimensions)

$$|23\rangle - |32\rangle\,, \qquad |31\rangle - |13\rangle\,, \qquad |12\rangle - |21\rangle\,.$$

Action of GL(3), e.g.

$$\begin{aligned} g(|12\rangle - |21\rangle) &= |ij\rangle(g_{i1}g_{j2} - g_{i2}g_{j1}) \\ &= \underbrace{|23\rangle(g_{21}g_{32} - g_{22}g_{31}) + |32\rangle(g_{31}g_{22} - g_{32}g_{21})}_{=(|23\rangle-|32\rangle)\det\begin{pmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{pmatrix}} \\ &\quad + \underbrace{|31\rangle(g_{31}g_{12} - g_{32}g_{11}) + |13\rangle(g_{11}g_{32} - g_{12}g_{31})}_{=(|31\rangle-|13\rangle)\,(-1)\det\begin{pmatrix} g_{11} & g_{12} \\ g_{31} & g_{32}\end{pmatrix}} \\ &\quad + \underbrace{|12\rangle(g_{11}g_{22} - g_{12}g_{21}) + |21\rangle(g_{21}g_{12} - g_{22}g_{11})}_{=(|12\rangle-|21\rangle)\det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}}\,, \end{aligned}$$

similarly for the other two basis elements. We find

$$\Gamma^{\square}(g) = \begin{pmatrix} \det\begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{21} & g_{23} \\ g_{31} & g_{33}\end{pmatrix} & \det\begin{pmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{pmatrix} \\ (-1)\det\begin{pmatrix} g_{12} & g_{13} \\ g_{32} & g_{33}\end{pmatrix} & \det\begin{pmatrix} g_{11} & g_{13} \\ g_{31} & g_{33}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{11} & g_{12} \\ g_{31} & g_{32}\end{pmatrix} \\ \det\begin{pmatrix} g_{12} & g_{13} \\ g_{21} & g_{23}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{11} & g_{13} \\ g_{21} & g_{23}\end{pmatrix} & \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} \end{pmatrix} = \operatorname{adj}(g)^T\,,$$

with the adjunct matrix $\mathrm{adj}(g)$. According to Cramer's rule $g^{-1} = \dfrac{\mathrm{adj}(g)}{\det g}$, i.e.

$$\Gamma^{\tiny\begin{matrix}\square\\[-4pt]\square\end{matrix}}(g) = \det g \cdot (g^{-1})^T \,.$$

Remark: This is true for arbitrary $N > 2$ and the Young diagram $\tiny\begin{matrix}\square\\[-4pt]\vdots\\[-4pt]\square\end{matrix}$ $(N-1$ boxes).

For SU(3) we have $\det g = 1$ and $g^{-1} = g^\dagger$, i.e. $\Gamma^{\tiny\begin{matrix}\square\\[-4pt]\square\end{matrix}}(g) = \overline{g}$. We write $\begin{matrix}\square\\[-4pt]\square\end{matrix} = \overline{\square}$ and also put a $\overline{\text{bar}}$ over the dimension

For $\mathrm{GL}(N)$, besides the defining rep g also $(g^{-1})^T$, $\overline{g}$ and $\overline{(g^{-1})^T}$ are N-dimensional irreps, in general non-equivalent.

For $\mathrm{SU}(N)$, due to $g^\dagger = g^{-1}$, we have

$$(g^{-1})^T = \overline{g} \qquad \text{and} \qquad \overline{(g^{-1})^T} = g \,,$$

i.e. at most two of the four irreps are non-equivalent. For SU(2), even g and $\overline{g}$ are equivalent, see Problem 40; for $N \geq 3$ they are are non-equivalent. In terms of Young diagrams we obtain the complex conjugate irrep by means of a simple procedure which we will study in the live session.