Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 2 (due on 5 May 2021)

Problem 5
Let G be a finite group acting on the set M; for $m \in M$ let $G_m = \{g \in G : gm = m\}$. Show:

a) For each $m \in M$ the set G_m is a subgroup of G.
b) If $n \in Gm$ then $G_n \cong G_m$.
c) $|Gm| \cdot |G_m| = |G|$ (orbit-stabiliser theorem).

Problem 6
Let W be the symmetry group of a cube. (We consider only rotations, no reflections.) Determine $|W|$, the order of W, by considering the action of W on corners, edges or faces of the cube and applying the orbit-stabiliser theorem.

Problem 7
Let G be a group. For every $g \in G$ conjugation with g is defined by the map $\hat{g} : G \to G$, $x \mapsto gxg^{-1}$. Show:

a) Conjugation defines an action, $(g, h) \mapsto \hat{g}(h)$, of G on itself.
b) G is abelian iff every orbit of this action has length one.

Problem 8
Let $\varphi : G \to H$ be a group homomorphism with kernel K and image B. Show:

a) K is a normal subgroup of G.
b) φ induces an isomorphism $\hat{\varphi} : G/K \to B$.

Problem 9
Let $\phi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{O}(3, 1)$ be the homomorphism to the Lorentz group, as introduced in the lectures. Let $\alpha, \beta \in [0, 2\pi]$, $r > 0$ and

$$U = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad V = \begin{pmatrix} \mathrm{e}^{-i\beta} & 0 \\ 0 & \mathrm{e}^{i\beta} \end{pmatrix}, \qquad B = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}.$$

Show:

a) $\phi(U)$ is a rotation about the x_2-axis by an angle 2α.
b) $\phi(V)$ is a rotation about the x_3-axis by an angle 2β.
c) $\phi(B)$ is a boost in x_3-direction, i.e.

$$\phi(B) = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}$$

for some $t \in \mathbb{R}$.