Universität Tübingen, Fachbereich Mathematik — Sommersemester 2021
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Groups and Representations

Homework Assignment 4 (due on 19 May 2021)

Problem 15 (Continuation of Problem 14)
We now determine all irreducible representations of D_4 (up to equivalence):

d) What are the dimensions of the irreducible representations?

e) Find all one dimensional irreducible representations.
Hint: First consider irreducible representations of quotient groups, cf. the remarks on (non-)faithful representations in Section 2.1.

f) Determine the character table and the remaining representation(s).

Problem 16
Let G be a finite group, $|G| = n$. We number the group elements, $G = \{g_j, j = 1 \dots n\}$, denote by m the number of conjugacy classes c (with n_c elements) and by p the number of non-equivalent irreducible representations Γ^i of G (with dimensions d_i).

Consider the matrix U with entries $u_{ja} = \sqrt{\frac{d_{i_a}}{n}}\, \Gamma^{i_a}(g_j)_{\mu_a \nu_a}$ with a triple $a = (i_a, \mu_a, \nu_a)$.

Employ the results of Sections 2.5 and 2.6 in order to solve the following problems.

a) Determine the dimensions of U and express the orthogonality relation for irreducible representations (Theorem 6) in terms of U.

b) Show:

(i) $\sum\limits_{i \le p} d_i \operatorname{tr}\left(\Gamma^i(g_j)\Gamma^i(g_k)^\dagger\right) = n\delta_{jk}$,

(ii) $\sum\limits_{g \in c} d_i\, \Gamma^i(g) = n_c\, \chi^i_c\, \mathbb{1}$ and

(iii) $\sum\limits_{i \le p} n_c\, \chi^i_c\, \overline{\chi^i_{c'}} = n\delta_{cc'}$.

c) Conclude that $m = p$.

Problem 17
Let V be a finite-dimensional vector space and $P : V \to V$ a linear operator with $P^2 = P$.

a) Show that there exist subspaces U and W with $V = U \oplus W$, $P|_U = \mathbb{1}$ and $P|_W = 0$.

Let $\langle \cdot, \cdot \rangle$ be a scalar product on V, and let $P^\dagger = P$.

b) Show that $U = W^\perp$.

Problem 18
Three spin-$\frac{1}{2}$ particles[1] define a representation D of S_3 on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^8$ by permutations of the particles, i.e. e.g. $D\left((12)\right)|{\uparrow\downarrow\uparrow}\rangle = |{\downarrow\uparrow\uparrow}\rangle$.

Which irreducible representations of S_3 are contained in D and how often does each of them appear?

Problem 19
Let $g = \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix}$, $u, v \in \mathbb{C}$ with $|u|^2 + |v|^2 = 1$.

a) Verify that $g \in \mathrm{SU}(2)$, and explain why every $g \in \mathrm{SU}(2)$ can be written in this way.

The basis vectors $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ of $\mathbb{C}^2$, as defined in the lecture[1], transform in the two-dimensional representation $\Gamma^2(g) = g \;\forall\, g \in \mathrm{SU}(2)$.

b) Write $\Gamma^2(g)|{\uparrow}\rangle$ and $\Gamma^2(g)|{\downarrow}\rangle$ as linear combinations of $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$.

Consider now $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the product basis $|{\uparrow\uparrow}\rangle = |{\uparrow}\rangle \otimes |{\uparrow}\rangle$ etc. (cf. lecture). Under $\mathrm{SU}(2)$ this basis transforms in $\Gamma^{2\otimes 2} = \Gamma^2 \otimes \Gamma^2$.

c) Expand $\Gamma^{2\otimes 2}|{\uparrow\uparrow}\rangle$ etc. in the product basis.

d) Show: $\mathrm{span}(|0,0\rangle)$ and $\mathrm{span}(|1,1\rangle, |1,0\rangle, |1,-1\rangle)$ (as defined in the lecture) are invariant under $\mathrm{SU}(2)$, and thus carry one- and three-dimensional representations of $\mathrm{SU}(2)$, respectively, i.e. $\Gamma^{2\otimes 2} = \Gamma^1 \oplus \Gamma^3$.

e) Explicitly determine the representation matrices $\Gamma^1(g)$ and $\Gamma^3(g)$.

On $\mathbb{C}^2 \otimes \mathbb{C}^2$ also acts – as in Problem 18 – a representation D of $S_2 \cong \mathbb{Z}_2 = \{e, (12)\}$.

f) In which representations of S_2 do the vectors $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$ and $|0,0\rangle$ transform?

[1] If "spin-$\frac{1}{2}$ particle" doesn't mean much to you, then just ignore the word. We introduced this manner of speaking in Section 2.8, and the only thing you need to know for this homework assignment are the definitions

$$|{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |{\downarrow\uparrow}\rangle = |{\downarrow}\rangle \otimes |{\uparrow}\rangle \quad \text{etc.}$$