Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Bernanda Telalovic Sommersemester 2021

Groups and Representations

Homework Assignment 6 (due on 9 June 2021)

Problem 22
We consider once more the CO_2 molecule of Problems 12 & 21. In Problem 12 we found a six-dimensional representation of V_4. Decompose the six-dimensional carrier space into irreducible invariant subspaces by applying the generalised projection operators.

Problem 23
$V = \mathbb{C}^2$ carries the 2-dimensional irreducible representation of $D_3 \cong S_3$ (cf. Section 2.4.1). On $W = V \otimes V$ we consider the corresponding product representation. Decompose W into irreducible invariant subspaces by applying the generalised projection operators, and determine the Clebsch-Gordan coefficients.

Problem 24
We consider the abelian group $C_3 = \{e, a, a^{-1}\} \cong \mathbb{Z}_3$.

a) How many (non-equivalent) irreps does C_3 have, what are their dimensions and how often do they appear in the regular rep?

b) Show that
$$e_1 = \frac{1}{3}(e + a + a^{-1})$$
is a primitive idempotent, generating the trivial rep.

c) Use the ansatz
$$e_2 = xe + ya + za^{-1}$$
in order to find all primitive idempotents.

d) For each primitive idempotent find out whether it generates a new (non-equivalent) irrep or an irrep equivalent to one generated by a previous idempotent.

e) Specify all minimal left ideals and construct the corresponding irreps of C_3. Collect your results in a table.

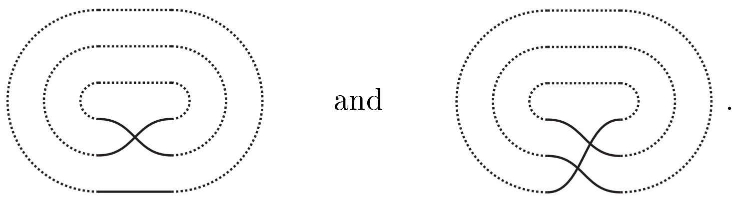
Problem 25

For $\sigma \in S_n$ and $j = 1, \ldots, n$ let $k_j(\sigma)$ be the number of (disjoint) cycles of length j in σ, e.g. $k_1(e) = n$ and $k_j(e) = 0 \; \forall \, j > 1$. Show:

a) The conjugacy class of σ is determined by its cycle structure, i.e.

$$[\sigma] := \{\tau\sigma\tau^{-1} : \tau \in S_n\} = \{\tau \in S_n : k_j(\tau) = k_j(\sigma), j = 1, \ldots, n\}\,.$$

It's almost trivial using the birdtrack notation (see Section 1.4)!
HINT: In order to make the cycle structure visible consider the birdtrack diagram of σ and connect the first line on the left to the first line on the right etc.; e.g. for $(12), (132) \in S_3$ consider



b) The number of elements of a class is given by

$$|[\sigma]| = \frac{n!}{\prod\limits_{j \le n} k_j! j^{k_j}}\,.$$

This is the perfect time for revisiting fun exercise (5) from Instruction 2.