Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Bernanda Telalovic Sommersemester 2021

Groups and Representations

Homework Assignment 7 (due on 16 June 2021)

Problem 26
Determine once more the characters of the irreps of S_3 by using the methods of Section 4.3.1.

Problem 27
Let V be a vector space and $A : V \to V$ a linear map. Show that if A is nilpotent (i.e. if for some $n \in \mathbb{N}$ we have $A^n v = 0 \ \forall\, v \in V$) then $\operatorname{tr} A = 0$.

Problem 28
In the quark model baryons are made out of three quarks. The latter are characterised i.a. by the quantum numbers I (isospin) and Y (hyper charge).

We have $(I, Y) = \left(\frac{1}{2}, \frac{1}{3}\right)$ for the up-quark, $|u\rangle$, $(I, Y) = \left(-\frac{1}{2}, \frac{1}{3}\right)$ for the down-quark, $|d\rangle$, and $(I, Y) = \left(0, -\frac{2}{3}\right)$ for the strange-quark, $|s\rangle$. For products like $|udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$ the values of I and Y are given by the sums of the values for the individual quarks.

For combinations of 3 quarks we thus have a 27-dimensional space V, which carries a representation of S_3 (by permutation of the factors).

a) Which irreps are contained in this representation and what are their multiplicities?

b) Let $U \subset V$ be an irreducible invariant subspace. What can we say about the values of I and Y on U?

c) In a (I, Y)-diagram mark all points corresponding to vectors transforming in the irrep defined by $\yng(3)$.

d) Repeat part (c) for the irrep with Young diagram $\yng(2,1)$. You find some potentially useful `Octave`/Matlab-Code on the course webpage.

Problem 29

For $A \in \mathbb{C}^{n\times n}$ the matrix exponential is defined as

$$\mathrm{e}^A = \exp(A) = \sum_{\nu=0}^{\infty} \frac{A^\nu}{\nu!}\,.$$

Prove:

a) The series converges absolutely and uniformly.
HINT: On $\mathbb{C}^{n\times n}$ use the operator norm

$$\|A\| = \sup_{v\in\mathbb{C}^n\setminus\{0\}} \frac{|Av|}{|v|}\,,$$

for which we have $\|AB\| \leq \|A\|\,\|B\|$.

b) For $T \in \mathrm{GL}(n)$ we have $\mathrm{e}^{TAT^{-1}} = T\mathrm{e}^A T^{-1}$.

c) e^{tA} is the unique solution of the initial value problem $\dot{X}(t) = AX(t)$, $X(0) = \mathbb{1}$.

d) For $t, s \in \mathbb{C}$ we have $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$.

e) $\left(\mathrm{e}^A\right)^\dagger = \mathrm{e}^{(A^\dagger)}$.

f) $\det \mathrm{e}^A = \mathrm{e}^{\operatorname{tr} A}$.