Universität Tübingen, Fachbereich Mathematik — Sommersemester 2021
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Groups and Representations

Homework Assignment 8 (due on 23 June 2021)

Problem 30

For a fixed partition λ of $n \in \mathbb{N}$ we define an ordering of standard tableaux as follows. Consider the sequence $(r^p_\lambda)_j$, $j = 1, \ldots, n$, of numbers in the boxes of Θ^p_λ starting with the first row read from left to right, then the second row from left to right etc. We say that $\Theta^p_\lambda > \Theta^q_\lambda$ if the first non-vanishing term in the sequence $(r^p_\lambda)_j - (r^q_\lambda)_j$, $j = 1, \ldots, n$, is positive. Then, e.g., the standard tableaux for $\lambda = (3, 2)$ are ordered as

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\,.$$

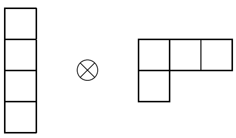
a) Prove that $e^p_\lambda e^q_\lambda = 0$ if $\Theta^p_\lambda > \Theta^q_\lambda$.

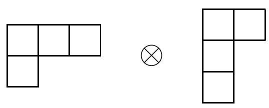
b) Show that (a) implies that the left ideals generated by the standard tablaux for a fixed partition are linearly independent.

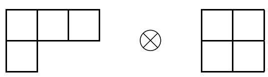
Problem 31

a) Determine the character table of S_4 using the methods of Section 5.5 (recursive method).

b) Consider the following product representations of S_4, determine which irreps they contain and how many times. Also write their decomposition into irreps in terms of Young diagrams.

(i) 

(ii) 

(iii) 

(iv) 

Problem 32
Let $\vec{n} \in S^2 \hookrightarrow \mathbb{R}^3$ be a unit vector in $\mathbb{R}^3$ and $\varphi \in \mathbb{R}$. We denote by σ_j, $j = 1, 2, 3$, the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and we define

$$\vec{\sigma} = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} .$$

Show that

$$\exp\left(-\mathrm{i}\tfrac{\varphi}{2}\vec{\sigma}\cdot\vec{n}\right) = \mathbb{1}\cos\tfrac{\varphi}{2} - \mathrm{i}\vec{\sigma}\cdot\vec{n}\sin\tfrac{\varphi}{2} ,$$

and verify that $\exp\left(-\mathrm{i}\frac{\varphi}{2}\vec{\sigma}\cdot\vec{n}\right) \in \mathrm{SU}(2)$.

HINT: First calculate $(\vec{\sigma}\cdot\vec{n})^2$.

Problem 33
We define $\mathfrak{sl}(2, \mathbb{C}) := \{A \in \mathbb{C}^{2\times 2} : \operatorname{tr} A = 0\}$. Then the matrix exponential is a map

$$\exp : \mathfrak{sl}(2, \mathbb{C}) \to \mathrm{SL}(2, \mathbb{C}) = \{B \in \mathbb{C}^{2\times 2} : \det B = 1\} .$$

a) Show that the matrix

$$S_a = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

is in the image of exp iff $a = 0$.

b) Is $\mathrm{SL}(2, \mathbb{C})$ compact?