Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Bernanda Telalovic

Sommersemester 2021

Groups and Representations

Homework Assignment 10 (due on 7 July 2021)

Problem 37

a) Determine the Haar measure for SU(2) in axis-angle parametrisation,

$$U = \exp\left(-\mathrm{i}\tfrac{\alpha}{2}\vec{\sigma}\cdot\vec{x}\right)\,,$$

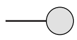
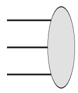
with $0 \leq \alpha \leq 2\pi$ and $\vec{x} \in S^2 \hookrightarrow \mathbb{R}^3$. Normalise s.t. $\mathrm{vol}(\mathrm{SU}(2)) = 1$.

Hint: It is convenient to first show $(\vec{x}\cdot\vec{\sigma})(\vec{y}\cdot\vec{\sigma}) = \mathbb{1}\vec{x}\cdot\vec{y} + \mathrm{i}\vec{\sigma}(\vec{x}\times\vec{y})$ and to use the unit vectors $\vec{e}_r, \vec{e}_\vartheta, \vec{e}_\varphi$ for spherical coordinates.

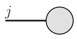
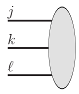
b) Use the result of (a) together with the results of Problem 35 in order to determine the Haar measure for SO(3) in the axis-angle parametrisation.

Problem 38

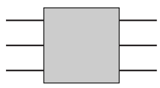
In birdtrack notation we write vectors $v \in V$ or tensors $t \in V^{\otimes n}$, as blobs with one or several lines attached, e.g.

$v =$  $\in V$ or $t =$  $\in V^{\otimes 3}$.

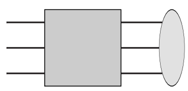
Expressed in a basis, components are

$v_j =$  or $t_{jk\ell} =$  ,

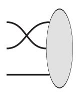
i.e. we write indices on the lines. Linear maps $A : V^{\otimes n} \to V^{\otimes n}$ are represented by blobs with n legs on each side, e.g.

$A =$  $: V^{\otimes 3} \to V^{\otimes 3}$,

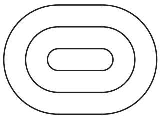
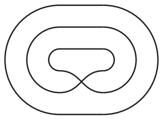
and with $t \in V^{\otimes 3}$ we have

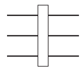
$At =$  $\in V^{\otimes 3}$.

Birdtracks for permutations $p \in S_n$, or $\in \mathcal{A}(S_n)$, see Problem 25 & Instruction 12, can now be applied (as linear maps) to elements of $V^{\otimes n}$, e.g.

$(12)t =$  .

We obtain traces of linear maps by connecting the first line on the left to the first line on the right etc. (cf. also Problem 25), with each loop contributing a factor of $\dim V = N$ (why?), e.g. for $e, (12) \in \mathcal{A}(S_3)$ we get

$\operatorname{tr} e =$  $= N^3$ and $\operatorname{tr}(12) =$  $= N^2$

a) Calculate the trace of $(132) \in S_3$ and the trace of  $\in \mathcal{A}(S_3)$.

b) Normalise the Young operators $e_{(3)}, e_{(2,1)}, e_{(2,1)}^{(23)}, e_{(1,1,1)} \in \mathcal{A}(S_3)$ of Instruction 13 s.t. they are idempotent. Determine the traces of these primitive idempotents. Later we will see that these are the dimensions of $\mathrm{GL}(N)$ irreps contained in $V^{\otimes 3}$.

HINT: Some identities from Instruction 12 are useful.