Sommersemester 2021

Groups and Representations*

Stefan Keppeler Universität Tübingen, Fachbereich Mathematik
Auf der Morgenstelle 10
72076 Tübingen, Germany

office: C4P31
email: `stefan.keppeler@uni-tuebingen.de`

course webpage: `www.tinyurl.com/ss21-grar`

*These lecture notes may be updated over the course of the lecture. I'm sure there are typos and more serious errors. When you spot one, please let me know. *Stefan Keppeler*

Contents

1 Introduction **4**
1.1 Why groups? Why representations? 4
1.2 Basic notions 5
1.3 Examples & outlook 7
1.4 Permutations – the symmetric group 9
1.5 Group actions 10
1.6 Conjugacy classes and normal subgroups 12
1.7 Cosets and quotient groups 14
1.8 Direct product 16
1.9 Example: The homomorphism from $\mathrm{SL}(2, \mathbb{C})$ to the Lorentz group 17

2 Representations **20**
2.1 Definitions 20
2.2 Equivalent Representations 21
2.3 Beispiele und Invariante Unterräume 23
2.4 Irreducible Representations 26
2.4.1 Example: O_A operators for the group D_3 28
2.5 Schur's Lemmas and orthogonality of irreducible representations 31
2.6 Characters 33
2.7 The regular representation 36
2.8 Product representations and Clebsch-Gordan coefficients 39

3 Applications in quantum mechanics **44**
3.1 Expansion in irreducible basis functions and selections rules 44
3.2 Invariance of the Hamiltonian and degeneracies 45
3.3 Perturbation theory and lifting of degeneracies 48

4 Expansion into irreducible basis vectors **51**
4.1 Projection operators onto irreducible bases 51
4.2 Irreducible operators and the Wigner-Eckart Theorem 55
4.3 Left ideals and idempotents 57
4.3.1 Dimensions and characters of the irreducible representations 61

5 Representations of the symmetric group and Young diagrams **63**
5.1 One-dimensional irreps and associate reps of S_n 63
5.2 Young diagrams and Young tableaux 65
5.3 Young operators 67
5.4 Irreducible representations of S_n 69
5.5 Calculating characters using Young diagrams 72

6 Lie groups **75**
6.1 Topological groups 75

6.2 Example: SO(2) . . . 78
6.3 Lie groups . . . 80
6.4 Lie algebras . . . 82
6.5 More on SO(3) . . . 85
6.6 Invariant integration: Haar measure . . . 88
6.6.1 Calculating the Haar measure for a Lie group . . . 89
6.7 Properties of compact Lie groups . . . 92
6.8 Irreducible representations of SO(3) . . . 94
6.9 Remarks on some classical Lie groups . . . 98
6.10 More on Lie algebras and related topics . . . 100

7 Tensor method for constructing irreducible representations of GL(N) and subgroups 103
7.1 Setting . . . 103
7.2 Decomposition of $V^{\otimes n}$ into irreducible subspaces with respect to S_n and GL(N) . . . 104
7.2.1 Symmetry classes . . . 104
7.2.2 Totally symmetric and totally anti-symmetric tensors . . . 105
7.2.3 Tensors with mixed symmetry . . . 106
7.2.4 Complete reduction of $V^{\otimes n}$ . . . 108
7.2.5 Dimensions of the GL(N)-representations . . . 109
7.3 Irreducible representations of U(N) and SU(N) . . . 112
7.4 Reducing tensor products in terms of Young diagrams . . . 113
7.5 Complex conjugate representations . . . 115

8 Applications in particle physics 118
8.1 Elementary particles . . . 118
8.2 SU(2) isospin . . . 119
8.3 SU(2) flavour . . . 120
8.4 SU(3) flavour and the quark model . . . 122
8.5 Gell-Mann-Okubo formula . . . 128

6 Lie groups (continued) 129
6.11 Roots and weights . . . 129
6.12 From roots to the classification of semi-simple Lie algebras . . . 133

1 Introduction

1.1 Why groups? Why representations?

Groups are
. . . ubiquitous,
. . . come in many different guises.

In this course: mainly finite groups & compact Lie groups.
(There's much more, but our selection is not only interesting in its own right, it's also a good starting point.)

Representations (reps)
. . . (very roughly) study groups using vector spaces (linearity!),
. . . convenient,
. . . in this course mostly vector spaces over $\mathbb{C}$, sometimes over $\mathbb{R}$, probably never over finite fields (again this is a good starting point for everything else),
. . . tell us something about the group in question,
. . . are how groups often show up in applications, e.g. in physics (quantum mechanics, atomic energy levels, selection rules, masses in particle physics,. . .).

Course plan (very roughly)
. . . develop rather complete theory for reps of finite groups (on complex vector spaces),
. . . study symmetric groups (and reps) in some details,
. . . see what we can carry over / what is new for (compact) Lie groups.

1.2 Basic notions

Definition: (group)
Let $G \neq \emptyset$ be a set and $\circ$ an operation $\circ : G \times G \to G$. We call $(G, \circ)$ a group if:

(G1) $a, b \in G \Rightarrow a \circ b \in G$ (closure)
(already implied by $\circ : G \times G \to G$)

(G2) $(a \circ b) \circ c = a \circ (b \circ c)\ \forall\ a, b, c \in G$ (associativity)

(G3) $\exists\ e \in G$ with $a \circ e = a = e \circ a\ \forall\ a \in G$ (identity / neutral element)

(G4) for each $a \in G\ \exists\ a^{-1} \in G$ with $a \circ a^{-1} = e = a^{-1} \circ a$, with e from (G3) (inverses)

If it is clear from the context which operation we talk about, then we often just write G instead of $(G, \circ)$.

Definition: (abelian group)
A group $(G, \circ)$ is called commutative or abelian, if in addition we have:
(G5) $a \circ b = b \circ a\ \forall\ a, b \in G$ (commutativity)

Remarks:

1. The identity e is unique.
2. For each $a \in G$ the corresponding inverse is unique.
3. Often we call the operation multiplication (or group multiplication) and write $a \cdot b$ or just ab instead of $a \circ b$.
4. If the number of group elements is finite, we speak of a *finite group*, and we call the number of group elements the *order* $|G|$ of the group. (otherwise: *infinite group*).
5. A finite group (order n) is completely determined by its *group table* (or multiplication table) (with n^2 elements)

	e	a	b	c	$\cdots$
e	e	a	b	c	$\cdots$
a	a	a^2	ab	ac	$\cdots$
b	b	ba	b^2	bc	$\cdots$
c	c	ca	cb	c^2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fact: No two elements within one row (or column) can be the same. (see exercises)
This implies the *rearrangement lemma*: If one multiplies all elements of a group $\{e, a, b, c, \ldots\}$ by one of the elements, one obtains again all elements, in general in a different order.
In other words: Each row and each column in the group multiplication table contains each of the group elements exactly once.

Examples:

1. $(\mathbb{Z}, +)$: $e = 0$, $a^{-1} = -a$ for $a \in \mathbb{Z}$ (abelian); analogously $(\mathbb{R}, +)$ or $(\mathbb{C}, +)$
2. $(\mathbb{R} \setminus \{0\}, \cdot)$: $e = 1$, $x^{-1} = \frac{1}{x}$ for $x \in \mathbb{R}$ (abelian); analogously $(\mathbb{Q} \setminus \{0\}, \cdot)$ or $(\mathbb{C} \setminus \{0\}, \cdot)$
3. G: set of all symmetry operations (rotations, reflections, ...), which leave a certain object (atom, molecule, geometrical object[2], ...) invariant.
   $\circ$: subsequent application of operations.
   G can be finite (e.g. for a cube) or infinite (e.g. for a sphere) – in general non-abelian.

Definition: (subgroup)
Let $(G, \circ)$ be a group. A subset $H \subseteq G$, which satisfies (G1)–(G4) (with the same operation $\circ$), is called a subgroup of G.

Remarks:

1. Every group has two trivial subgroups: $\{e\}$ and G.
   All other subgroups are called non-trivial.
2. $|G|$ (if finite) is divisible by $|H|$. (will be proved later)

Definition: (homomorphism)
Given two groups $(G, \circ)$ and $(G', \bullet)$, a map $f : G \to G'$ is called a homomorphism, if

$$f(a \circ b) = f(a) \bullet f(b) \quad \forall\ a, b \in G\,.$$

Remarks:

1. A homomorphism f maps the identity to the identity and inverses to inverses, more precisely $f(e_G) = e_{G'}$ and $f(a^{-1}) = f(a)^{-1}\ \forall\ a \in G$.
2. The *image* of the homomorphism $f : G \to G'$ is

   $$\mathrm{im}(f) = f(G) = \{f(g) \,:\, g \in G\}\,,$$

   the *kernel* of f is the preimage of the identity of G' ,

   $$\ker(f) = \{g \in G \,:\, f(g) = e_{G'}\}\,.$$

Definition: (isomorphism)
A bijective homomorphism $f : G \to G'$ is called isomorphism. We then say that G and G' are isomorphic, and write $G \cong G'$.

Remark:

1. Isomorphic groups have the same group table, i.e. they are identical except for what we call their elements (and the group operation). (correspondingly for infinite groups)

[2]For a mattress (rectangle) we obtain the Klein four-group, see e.g. `https://opinionator.blogs.nytimes.com/2010/05/02/group-think/`

1.3 Examples & outlook

1. A group of the kind

$$\{\underbrace{e, a, a^2, \ldots, a^{n-1}}_{\text{pairwise different}}\}, \quad a^n = e\,,$$

is called *cyclic group* C_n
The smallest non-cyclic group is of order 4.
The smallest non-abelian group is of order 6.

2. A group with two elements: $\{e, a\}$
We have: $ee = e$, $ea = a$ and $ae = e$.
What about aa? ($= a$ or $= e$)
Group table:

	e	a
e	e	a
a	a	e

... only possibility since we cannot have an element twice in one row or column, (see above)
This is C_2. (see example 1)
$\Rightarrow$ Any group of order 2 is isomorphic to C_2;
in particular $C_2 \cong \mathbb{Z}_2 := (\{0, 1\}, + \bmod 2)$.

3. Examples for groups isomorphic to $\mathbb{Z}_2$:
   (a) Consider the following two maps $\mathbb{R}^n \to \mathbb{R}^n$,

$$e : \vec{x} \mapsto \vec{x}\,,$$
$$P : \vec{x} \mapsto -\vec{x} \quad (\text{parity})\,.$$

   group operation: composition of maps
   $\Rightarrow e \circ e = e$, $e \circ P = P$, $P \circ e = P$, $P \circ P = e$, i.e. isomorphic to $\mathbb{Z}_2$. (it has to)

   (b) Instead of the two spatial transformations consider now
   operators acting on (real- or complex-valued) functions f of $\vec{x}$:

$$(O_e f)(\vec{x}) = f(\vec{x})$$
$$(O_P f)(\vec{x}) = f(-\vec{x})$$

   $\Rightarrow O_e^2 = O_e$, $O_e O_P = O_P$, $O_P O_e = O_P$, $O_P^2 = O_e$, i.e. isomorphic to $\mathbb{Z}_2$.

   Remark: These operators are linear, i.e.

$$O(\alpha f + \beta g) = \alpha O(f) + \beta O(g)\,.$$

(c) Consider operators acting on complex-valued functions of two variables (physics: wave function of two particles)

$$(O_E\psi)(\vec{x}_1,\vec{x}_2) = \psi(\vec{x}_1,\vec{x}_2)$$
$$(O_S\psi)(\vec{x}_1,\vec{x}_2) = \psi(\vec{x}_2,\vec{x}_1)$$

$O_S^2 = O_E \ldots \;\Rightarrow \{O_E, O_S\} \cong \mathbb{Z}_2$
(different names than operators in example 3b in order to emphasise the different realisations)

When we will have learned about group actions and representations, we can revisit these examples from a different point of view, not just as homomorphisms.

$\mathbb{Z}_2$ looks rather innocent, but many concepts which we want to discuss in the following can already by illustrated for $\mathbb{Z}_2$.

4. Consider now example 3b and two functions f_e and f_o with

$$(O_P f_e)(\vec{x}) = f_e(\vec{x}) \qquad \text{“even parity”}$$
$$(O_P f_o)(\vec{x}) = -f_o(\vec{x}) \qquad \text{“odd parity”}$$

(e.g. $\vec{x} = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) \in \mathbb{R}^3,\ f_e(\vec{x}) = x^2 + yz,\ f_o(\vec{x}) = xy\sin z$)

f_e und f_o show a special behaviour under application of $\{O_e, O_P\}$:

- f_e is invariant under O_P
- f_o only changes the sign under O_P

Applications of group and representation theory in physics take advantage of the invariance of subspaces formed by even or odd functions, respectively; similarly for more complicated groups, as we will see later.

5. The identity (if integral exists)

$$\int_{\mathbb{R}^d} \overline{f_e(\vec{x})}\, f_o(\vec{x})\, \mathrm{d}^d x = 0$$

is an example for an “orthogonalty relation” between objects with special symmetry properties (“selection rule” in quantum mechanics; more later).

6. Any function can be written as a sum of an even and an odd function

$$f = f_e + f_o \quad \text{with} \quad f_e = \frac{1}{2}\big(f(\vec{x}) + f(-\vec{x})\big)$$
$$f_o = \frac{1}{2}\big(f(\vec{x}) - f(-\vec{x})\big)\,.$$

This illustrates that we can expand “objects” without special symmetry properties into linear combinations of “objects” with special symmetry properties.

1.4 Permutations – the symmetric group

Definition: (symmetric group)
The symmetric group of degree n, S_n, are the bijections of $\{1, 2, \ldots, n\}$ to itself under composition.

Remarks:

1. Elements of S_n are called permutations.
2. $|S_n| = n!$
3. two-line notation: write image of first line in second line, e.g.

$$S_6 \ni \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 1 & 2 & 5 & 3 \end{pmatrix}$$

   means $\pi(1) = 6$, $\pi(2) = 4$, ...

4. Every permutation can be written as a product of *disjoint cycles*, e.g.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 1 & 2 & 5 & 3 \end{pmatrix} = (163)(24)(5) \qquad \text{3-cycle, 2-cycle, 1-cycle}$$
$$= (163)(24) \qquad \text{usually omit 1-cycles}$$

   - where (163) means $\pi(1) = 6$, $\pi(6) = 3$, $\pi(3) = 1$, and thus

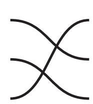
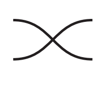
$$(163) = (631) = (316) \text{ but } \neq (136)\,.$$

   - Disjoint cycles commute, e.g. $(163)(24) = (24)(163)$.
   - Every ℓ-cycle ($\ell > 2$) can be written as a product of 2-cycles (transpositions), e.g.

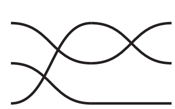
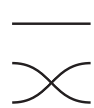
$$(163) = (13)(16)\,,$$

     where $(13)(16)$ is shorthand for $(13) \circ (16)$.

5. diagrammatic birdtrack notation: for $\pi \in S_n$ draw lines which end in position $1, \ldots, n$ on the right and in position $\pi(1), \ldots, \pi(n)$ on the left, e.g. $\pi, \sigma \in S_3$,

$\pi = (132) =$  , $\sigma = (12) =$  ,

   and for composition we compose diagrams and twist lines at will (it only matters where lines end),

$\pi\sigma = \pi \circ \sigma =$  $=$  .

Examples:

1. $S_2 = \{e, (12)\} \cong \mathbb{Z}_2$
2. $S_3 = \{e, (12), (13), (23), (123), (132)\}$
   - group table: see exercises
   - S_3 is non-abelian (the smallest non-abelian group), as are all S_n with $n \geq 3$, since e.g.
   $$(12)(13) = (132) \neq (13)(12) = (123)\,.$$
   - subgroups: $\{e\}$ and S_3 (trivial)
     $\{e, (12)\}$, $\{e, (13)\}$, $\{e, (23)\}$, all $\cong \mathbb{Z}_2$
     $\{e, (123), (321)\} \cong C_3$

Theorem 1. (Cayley)
Every group of order n is isomorphic to a subgroup of S_n.

Proof:
Write in a slightly unorthodox way by explicitly using properties of the group table – just to keep Problem 1 interesting.
Let $(G, \cdot)$ be a finite group, $|G| = n$. For $h \in G$ define

$$\begin{aligned} \varphi_h : G &\to G \\ g &\mapsto \varphi_h(g) = h \cdot g\,. \end{aligned}$$

φ_h permutes the n elements of G (since it yields a row of the group table). Now

$$\begin{aligned} f : g &\mapsto \varphi_g \\ G &\to G' := \{\varphi_g : g \in G\} \end{aligned}$$

is a homomorphism, because (i)

$$(\varphi_a \circ \varphi_b)(g) = \varphi_a(\varphi_b(g)) = \varphi_a(b \cdot g) = a \cdot b \cdot g = \varphi_{a \cdot b}(g)\,,$$

and because (ii) f is injective (otherwise there would be two equal lines in the group table of G), i.e. $G \cong G'$.
Further, G' contains only permutations of the n elements of G, i.e. G' is isomorphic to a subgroup of S_n. □

1.5 Group actions

Definition: (group action)
Let G be a group and M a set. A (group) action of G on M is a map

$$\begin{aligned} G \times M &\to M \\ (g, m) &\mapsto gm\,, \end{aligned}$$

which satisfies

$$em = m \quad \forall\, m \in M \qquad \text{and}$$
$$g(hm) = (gh)m \quad \forall\, g,h \in G \text{ and } \forall\, m \in M\,.$$

Remark: Thus, $M \to M$, $m \mapsto gm$, is bijective for each (fixed) $g \in G$, since $gm_1 = gm_2 \Rightarrow g^{-1}gm_1 = g^{-1}gm_2 \Leftrightarrow m_1 = m_2$ (injective) and $m \in M \Rightarrow gm' = m$ with $m' = g^{-1}m$ (surjective).

Definition: (orbit)
The orbit of the point $m \in M$ under an action of a group G on M is defined as

$$Gm = \{gm \,:\, g \in G\}\,.$$

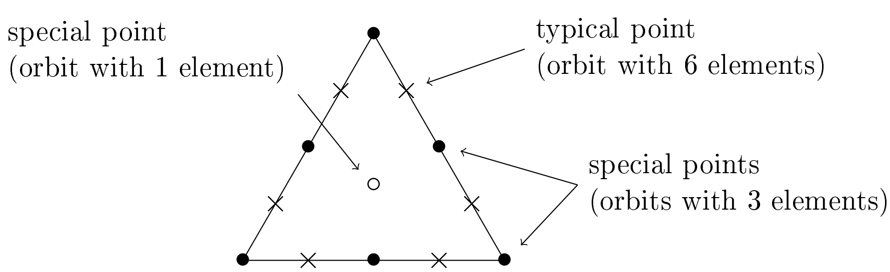
Remarks:

1. The orbit of a "typical" point contains $n = |G|$ elements.
2. The orbit of a "special" point contains less than $n = |G|$ elements.

Example:
Consider D_3, the symmetry group of an equilateral triangle ("D" for dihedral group).
$D_3 \cong S_3$ (permutations of the triangle's corners).

Group elements:
- identity
- 2 rotations (about 120° and 240°)
- 3 reflections (axes through each of the corners)

D_3 acts naturally on M, a plane with the origin in the centre of the triangle.



Definition: (stabiliser)
Let $G \times M \to M$, $(g,m) \mapsto gm$, be an action of G auf M. The set of group elements that map a given $m \in M$ to itself, i.e.

$$G_m = \{g \in G \,:\, gm = m\}\,,$$

is called stabiliser (or isotropy group or little group) of m.

Remark: G_m is a group (see exercises).

For the D_3-**example** (see above):

- the stabiliser of $\times$ ist $\{e\}$
- the stabiliser of $\circ$ ist D_3
- the stabiliser of $\bullet$ ist $\{I, \sigma\} \cong \mathbb{Z}_2$, where σ is the reflection across the axis though $\bullet$

Notice that in all three cases $|Gm| \cdot |G_m| = |G|$. This is true in general for finite groups (*orbit-stabiliser theorem*, see exercises).

1.6 Conjugacy classes and normal subgroups

Definition: (conjugation)
Let G be a group. We say $x \in G$ is conjugate to $y \in G \underset{\text{Def.}}{\Leftrightarrow} \exists\, g \in G \,:\; y = gxg^{-1}$.
We then write $x \sim y$.

Remark:
$\sim$ defines an equivalence relation, since

1. reflexivity: $x \sim x \;\forall\, x \in G$ (with $g = e$).
2. symmetry: $x \sim y \Leftrightarrow y \sim x$ (with $g \leftrightarrow g^{-1}$)
3. transitivity: $x \sim y$ und $y \sim z \Rightarrow x \sim z$ ($y = gxg^{-1}, z = hyh^{-1} \;\Rightarrow\; z = (hg)x(hg)^{-1}$)

Examples:

1. $G = S_3$: $(13) \sim (12)$, since $(23)(12)\underbrace{(23)^{-1}}_{=(23)} = (13)$
2. $G = \mathrm{SO}(3)$, group of spatial rotations in 3 dimensions:
   $R_{\vec{n}}(\phi)$ = rotation about axis $\vec{n}$ by angle ϕ
   For arbitrary $R \in \mathrm{SO}(3)$ we have $RR_{\vec{n}}(\phi)R^{-1} = R_{\vec{n}'}(\phi)$ with $\vec{n}' = R\vec{n}$, i.e. rotations by the same angle but about different axes are conjugate to each other.

Definition: (conjugacy class)
For a group G and $x \in G$ we call $\{gxg^{-1} \,:\, g \in G\}$ the conjugacy class of x.

Remarks:

1. The class of e contains only e, since $geg^{-1} = e \;\forall\, g$.
2. For abelian groups each element forms a class of its own, since $gxg^{-1} = x \;\forall\, g$.
3. In general a class is not a subgroup (cf. below).
4. Each element of G is contained in exactly one class, since it's an equivalence relation…transitivity.
5. $|G|$ is divisible by the number of elements of each conjugacy class. (orbit-stabiliser theorem, cf. exercises).
6. Later: The number of conjugacy classes is equal to the number of non-equivalent

irreducible representations of a group.

Example: S_3
First class: $\{e\}$.
Now conjugate (12) with all elements of S_3,

$$\begin{aligned} e(12)e &= (12) \\ (12)(12)(12) &= (12) \\ (13)(12)(13) &= (23) \\ (23)(12)(23) &= (13) \\ (123)(12)(132) &= (23) \\ (132)(12)(123) &= (13) \end{aligned}$$

i.e. (12), (13) and (23) form a class.
For the remaining two elements we have

$$(12)(123)(12) = (132)$$

i.e. $(123) \sim (132)$ and thus contained in the same class.
We found 3 classes:

$$C_e = \{e\}\,, \qquad C_{(12)} = \{(12),(13),(23)\}\,, \qquad C_{(123)} = \{(123),(321)\}\,.$$

Notice: Two elements of S_3 are conjugate if they have the same cycle structure; this is true for S_n in general (later).

For $D_3 \cong S_3$: $C_{(12)}$ – reflections , $C_{(123)}$ – rotations

Definition: (conjugate subgroups, normal subgroup)

(i) We call a subgroup $K \subseteq G$ conjugate to a subgroup $H \subseteq G$ if $\exists\, g \in G$ such that

$$K = gHg^{-1} = \{ghg^{-1} \,:\, h \in H\}\,.$$

(ii) If $ghg^{-1} \in H \;\forall\, h \in H$ und $\forall\, g \in G$ then we call H a normal subgroup (or invariant subgroup) of G.

Examples:

1. The subgroup $K = \{e,(13)\} \subset S_3$ is conjugate to $H = \{e,(12)\}$, since $(23)e(23)^{-1} = e$ und $(23)(12)(23)^{-1} = (13)$.
2. Every group has two trivial normal subgroups: $\{e\}$ and G.
3. The only non-trivial normal subgroup of S_3 is $\{e,(123),(132)\}$.

Remark: A finite group is called *simple* if it has no non-trivial subgroup.

Thus, S_3 is not simple.

1.7 Cosets and quotient groups

Definition: (coset)
Let G be a group and $H \subseteq G$ a subgroup. For $g \in G$ the set

$$gH := \{gh : h \in H\}$$

is called a left coset of H (in G). Similarly we call

$$Hg := \{hg : h \in H\}$$

a right coset of H.

Remarks:

1. $gH, Hg \subseteq G$.
2. If $g \in H \Rightarrow gH = Hg = H$ (rearrangement lemma, cf. Problem 1).
3. The number of elements of a coset is equal the order of the subgroup, shortly $|gH| = |H|$.
4. In the following we consider mostly left cosets.
5. Two cosets g_1H and g_2H are either identical ($\Leftrightarrow g_1^{-1}g_2 \in H$) or disjoint.
   Proof: Assume that there is a common element, i.e.
   $\exists\, h_1, h_2 \in H \,:\, g_1h_1 = g_2h_2$
   $\Leftrightarrow g_2 = g_1h_1h_2^{-1}$
   $\Rightarrow g_2H = g_1h_1h_2^{-1}H = g_1H$ $\square$
6. Since each $g \in G$ is element of exactly one coset, and since $|gH| = |H|$, it follows that H divides $|G|$ (cf. 1.2).[3]

Example:
For S_3: Let $H_1 = \{e, (12)\}$ (not normal) and $H_2 = \{e, (123), (132)\}$ (normal).

- Left and right cosets of H_1:

$$\begin{aligned} eH_1 &= \{e, (12)\} & H_1e &= \{e, (12)\} \\ (12)H_1 &= \{(12), e\} & H_1(12) &= \{(12), e\} \\ (13)H_1 &= \{(13), (123)\} & H_1(13) &= \{(13), (132)\} \\ (123)H_1 &= \{(123), (13)\} & H_1(132) &= \{(132), (13)\} \\ (23)H_1 &= \{(23), (132)\} & H_1(23) &= \{(23), (123)\} \\ (132)H_1 &= \{(132), (23)\} & H_1(123) &= \{(123), (23)\} \end{aligned}$$

  Left and right cosets are different, and, e.g.

$$S_3 = H_1 \cup (13)H_1 \cup (23)H_1\,.$$

[3] Alternatively, we could define an action of G on G by left multiplication and then invoke the orbit-stabiliser theorem.

- Cosets of H_2:

$$
\begin{aligned}
eH_2 &= \{e,(123),(132)\} & H_2e &= \{e,(123),(132)\}\\
(123)H_2 &= \{(123),(132),e\} & H_2(123) &= \{(123),(132),e\}\\
(132)H_2 &= \{(132),e,(123)\} & H_2(132) &= \{(132),e,(123)\}\\
(12)H_2 &= \{(12),(23),(13)\} & H_2(12) &= \{(12),(13),(23)\}\\
(13)H_2 &= \{(13),(12),(23)\} & H_2(13) &= \{(13),(23),(12)\}\\
(23)H_2 &= \{(23),(13),(12)\} & H_2(23) &= \{(23),(12),(13)\}
\end{aligned}
$$

  Left and right cosets are identical, and, e.g.

$$S_3 = H_2 \cup (12)H_2$$

Generally: If H is a normal subgroup of G then left and right cosets are identical, since

$$gHg^{-1} = H \quad \Leftrightarrow \quad gH = Hg\,.$$

Then the partitioning of G into cosets is unique.

If H is normal, then the cosets form a group...

Definition: (quotient group)
Let H be a normal subgroup of G. We define the quotient group $(G/H,\cdot)$ as the set of cosets,

$$G/H := \{gH \,:\, g\in G\}\,,$$

with the group law

$$(g_1H)\cdot(g_2H) = (g_1g_2)H\,.$$

Remarks:

1. $|G/H| = \frac{|G|}{|H|}$
2. $(G/H,\cdot)$ is actually a group, since
   (G1) $g_1, g_2 \in G \Rightarrow (g_1g_2)H \in G/H$,
   (G2) associativity of G carries over to G/H,
   (G3) $e_{G/H} = H$, because $gH\cdot H = gH = H\cdot gH$, and
   (G4) the inverse of gH is $g^{-1}H$, because $gH\cdot g^{-1}H = H = g^{-1}H\cdot gH$.
3. Where did we need that H is normal (d.h. $gHg^{-1} = H\ \forall\ g\in G$)? Otherwise, in general the group law $\cdot$ isn't a well-defined map $G/H\times G/H \to G/H$. Replacing H by hH with some $h\in H$ must not change the result, but

$$
\begin{aligned}
(g_1hH)\cdot(g_2H) = (g_1hg_2)H &\underset{\text{in general}}{\neq} (g_1g_2)H\\
&= (g_1g_2\,g_2^{-1}hg_2)H
\end{aligned}
$$

   However, if H is normal then $g_2^{-1}hg_2 \in H$ und thus $(g_1g_2\,g_2^{-1}hg_2)H = (g_1g_2)H$.

Examples:

- $H_2 = \{e, (123), (132)\} \subset S_3$ is normal. The quotient group S_3/H_2 has two elements,

$$\{e, (123), (132)\} \qquad \text{and} \qquad \{(12), (13), (23)\}$$

and is thus isomorphic to $\mathbb{Z}_2$.

- $H_1 = \{e, (12)\} \subset S_3$ is not normal, e.g. $(123)(12)(123)^{-1} = (23) \notin H_1$, and thus $\cdot$ is not well-defined, e.g.

$$\begin{aligned}(eH_1)((13)H_1) &= (13)H_1 = \{(13), (123)\} \\ &\neq ((12)H_1) \cdot ((13)H_1) = (12)(13)H_1 = (132)H_1 = \{(132), (23)\}\,.\end{aligned}$$

1.8 Direct product

Definition: (direct product)
For two groups $(A, \circ)$ and $(B, \bullet)$ the direct product is the Cartesian product $A \times B$ with group law

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \circ a_2, b_1 \bullet b_2)\,.$$

Remarks:

1. $e_{A\times B} = (e_A, e_B)$ and $(a, b)^{-1} = (a^{-1}, b^{-1})$.
2. For finite groups $|A \times B| = |A||B|$.
3. $G := A \times B$ has a normal subgroup isomorphic to A, namely

$$(A, e_B) := \{g \in G \,:\, g = (a, e_B) \text{ with } a \in A\}\,.$$

"normal" since for $a_1 \in A$ and $(a_2, b_2) \in G$ we have

$$g(a_1, e_B)g^{-1} = (a_2, b_2)(a_1, e_B)(a_2^{-1}, b_2^{-1}) = (a_2 a_1 a_2^{-1}, b_2 e_B b_2^{-1}) = (\underbrace{a_2 a_1 a_2^{-1}}_{\in A}, e_B)\,.$$

Similarly for B.
Furthermore $A \cong G/B$ (and vice versa):[4]

$$G/B = \{(a, b)B \,:\, (a, b) \in G\} = \{(a, B) \,:\, a \in A\} \qquad \text{(rearrangement lemma)}$$

Caveat: In general, for a normal subgroup H of G, $G \ncong H \times (G/H)$ (since in general G/H isn't a normal subgroup[5] of G).
Example: S_3 has subgroups $H_1 = \{e, (12)\}$ and $H_2 = \{e, (123), (132)\}$.
H_2 is normal.
$S_3/H_2 \cong \mathbb{Z}_2 \cong H_1$, but $S_3 \ncong H_1 \times H_2$, since H_1 isn't a normal subgroup, or, in other words, the elements of H_1 und H_2 don't commute.

[4] here B is shorthand for (e_A, B)
[5] In general G/H doesn't even need to be isomorphic to a subgroup of G.

1.9 Example: The homomorphism from SL(2, ℂ) to the Lorentz group

- Let M be the Minkowski space, i.e. $M = \mathbb{R}^4$ with the Lorentz metric[6]
$$\|x\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \,.$$
We call $x = (x_0, x_1, x_2, x_3)$ a four-vector.

- A (homogeneous) Lorentz transformation Λ is a linear map $M \to M$, which preserves the Lorentz metric, i.e.
$$\|\Lambda x\|^2 = \|x\|^2 \qquad \forall\, x \in M \,.$$

- The Lorentz group $L = \mathrm{O}(3,1)$ is the group of all (homogeneous) Lorentz transformations.

- Identify each $x \in M$ with a Hermitian 2×2 matrix:[7]
$$X := f(x) := x_0 \mathbb{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \qquad \text{with}$$
$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\text{i.e.} \quad X = \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix}$$
The σ_j are called Pauli matrices. It follows that
$$\det X = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \|x\|^2 \,.$$

- Let now $A \in \mathrm{GL}(2,\mathbb{C}) := \{B \in \mathbb{C}^{2\times 2} \,:\, \det B \neq 0\}$ (group under matrix multiplication). Define an action of $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{C}^{2\times 2}$ by
$$\mathbb{C}^{2\times 2} \ni X \mapsto AXA^\dagger$$
and denote the induced action on M by
$$M \ni x \mapsto \phi(A)x := f^{-1}(Af(x)A^\dagger) \,.$$

- We have $(AXA^\dagger)^\dagger = AXA^\dagger$, i.e. $AXA^\dagger$ is Hermitian and thus $\phi(A)x$ is a (real) four-vector. Furthermore,
$$\|\phi(A)x\|^2 = \det(AXA^\dagger) = |\det A|^2 \det X = |\det A|^2 \|x\|^2 \,.$$

- With $A \in \mathrm{SL}(2,\mathbb{C}) := \{B \in \mathbb{C}^{2\times 2} \,:\, \det B = 1\}$ we have
$$\|\phi(A)x\|^2 = \|x\|^2 \,,$$
i.e. $\phi(A)$ corresponds to Lorentz transformation.

[6] more precisely $\|x\|^2 = d(x,x)$ with the pseudo-Riemannian metric $d(x,y) = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$.

[7] The Hermitian 2×2 matrices form a (real) four-dimensional vector space, a basis of which is given by $\mathbb{1}$ and the Pauli matrices.

- Furthermore,

$$\phi(A)\phi(B)x = \phi(A)f^{-1}(Bf(x)B^\dagger) = f^{-1}(ABf(x)B^\dagger A^\dagger) = \phi(AB)x\,,$$

  i.e. $\phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(3,1)$ is a group homomorphism.

- ϕ is no isomorphism, since $\phi(-A) = \phi(A)$ (not injective).

- **Examples** (see exercises):
  1. For the matrix

$$U_\theta = \begin{pmatrix} e^{-\mathrm{i}\theta} & 0 \\ 0 & e^{\mathrm{i}\theta} \end{pmatrix}$$

     $\phi(U_\theta)$ is a rotation about the x_3-axis by the angle 2θ.
  2. For the matrix

$$V_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

     $\phi(V_\alpha)$ is a rotation about the x_2-axis by the angle 2α.
  3. For the matrix

$$M_r = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$$

     $\phi(M_r)$ is a Lorentz boost in x_3-direction with parameter $2\ln(r)$.
     By the way: The boosts alone (in arbitrary directions) do not form a group.

The homomorphism $\phi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{O}(3,1)$ isn't surjective either:

- $\mathrm{SL}(2,\mathbb{C})$ is (path-)connected (without proof).

- $\mathrm{O}(3,1)$ is disconnected (four connected components).

  - proper Lorentz transformations: $\det\Lambda = +1$
    improper Lorentz transformations: $\det\Lambda = -1$

  - orthochronous (time direction preserving) Lorentz transformations: $\Lambda_{00} \geq 1$
    non-orthochronous Lorentz transformations: $\Lambda_{00} \leq -1$

  - only the proper, orthochronous Lorentz transformations are in the same connected component as e. They form the subgroup L^0.

- $\mathrm{im}(\phi) = L^0$ (cf. exercises).

Homomorphism from SU(2) to O(3)

- SU(2) is the group of unitary 2×2 matrices with unit determinant 1, i.e. $\mathrm{SU}(2) := \{A \in \mathbb{C}^{2\times 2} \,:\, AA^\dagger = \mathbb{1} \text{ and } \det A = 1\} \subset \mathrm{SL}(2,\mathbb{C})$.
- How does $A \in \mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$ act on $e_0 = (1,0,0,0)$?
  $E_0 := f(e_0) = \mathbb{1}$ and thus
  $$E_0 \to AE_0A^\dagger = A\mathbb{1}A^\dagger = \mathbb{1} = E_0 \qquad \text{i.e.} \quad \phi(A)e_0 = e_0.$$
- $\mathrm{O}(3) := \{R \in \mathbb{R}^{3\times 3} \,:\, RR^T = \mathbb{1}\}$ is the group of orthogonal 3×3 matrices.
- For a Lorentz transformation of the form
  $$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \quad \text{with } R \in \mathrm{O}(3)$$
  we have $\Lambda e_0 = e_0$ (and vice versa), i.e. these transformations form a subgroup of $\mathrm{O}(3,1)$ which is isomorphic to O(3).[8]
  Thus, ϕ is also a homomorphism $\mathrm{SU}(2) \to \mathrm{O}(3)$.
  - It is once more 2-to-1, since $\phi(A) = \phi(-A)$.
  - Similar to the analysis above, $A \in \mathrm{SU}(2)$ is mapped to such $\phi(A) \in \mathrm{O}(3)$ which lie in the connected component of $\mathbb{1}$, i.e. those with determinant 1, i.e. $\phi(\mathrm{SU}(2)) = \mathrm{SO}(3)$.

[8]One also says: O(3) is a subgroup of $\mathrm{O}(3,1)$.

2 Representations

We will rarely, if ever, fix an explicit basis, but thinking this way makes it easier to manipulate tensorial objects.

Predrag Cvitanović

2.1 Definitions

Definition: (representation)
Let G be a group and V a vector space. A representation (rep) Γ of G is a homomorphism $G \to \mathrm{GL}(V)$, i.e. into the bijective linear maps $V \to V$, i.e. in particular

$$\Gamma(g)\Gamma(h) = \Gamma(gh) \quad \forall\, g, h \in G$$

and $\Gamma(e) = \mathbb{1}$ (identity matrix/operator). We call $\dim V$ the dimension of the representation, and we will require $\dim V > 0$.

Remarks:

1. A representation is an action of G on V (in addition: linear).
2. We say that V carries the representation Γ, and we call V the carrier space (of Γ).
3. Unless otherwise stated we consider vector spaces over $\mathbb{C}$ (maybe sometimes over $\mathbb{R}$, probably never over other fields),
   e.g. $\mathbb{C}^n$ or $L^2(\mathbb{R}^d)$,[9]
   equipped with a scalar product $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$, i.e. with $\forall\, v, w \in V$ and $\forall\, \alpha \in \mathbb{C}$:
   (i) $\langle v|w\rangle = \overline{\langle w|v\rangle}$
   (ii) $\langle v|\alpha w\rangle = \alpha\langle v|w\rangle$
   (iii) $\langle v|v\rangle \geq 0$ and $= 0$ only for $v = 0$
4. Choosing an orthonormal basis of V (if finite-dimensional), i.e. $\{v_j : j = 1, \ldots, \lambda = \dim V\}$, then each $\Gamma(g)$ corresponds to a $\lambda \times \lambda$ matrix with elements

   $$\Gamma(g)_{jk} = \langle v_j|\Gamma(g)v_k\rangle\,,$$

   and we call Γ a *matrix representation.*

   We say: The v_i transform under G in the representation Γ.
5. If V is a finite-dimensional vector space over $\mathbb{C}$, then $V \cong \mathbb{C}^{\dim V}$ and $\dim V = \operatorname{tr}\Gamma(e)$.

Definition: (faithful representation)
We call a representation faithful if the homomorphism $\Gamma : G \to \mathrm{GL}(V)$ is injective, i.e. different group elements are represented by different matrices.

[9] It's best to think of the finite-dimensional case for the moment. In the infinite-dimensional case we'd really want separable Hilbert spaces and bounded linear operators $\Gamma(g)$.

Remarks:

1. Every group has the trivial representation, with $\Gamma(g) = \mathbb{1} \; \forall \; g \in G$; in general not faithful.
2. If the group G has a non-trivial normal subgroup H, then a representation of the quotient group G/H also induces a representation of G. This representation is not faithful. (cf. Problem 9)

   Idea: $\tilde{\Gamma}(g) := \Gamma(gH) \Rightarrow$ (i) $\tilde{\Gamma}(g)\tilde{\Gamma}(h) = \Gamma(gH)\Gamma(hH) = \Gamma(ghH) = \tilde{\Gamma}(gh)$,
   (ii) $\Gamma(h) = \mathbb{1} \; \forall \; h \in H$.

   Conversely: If a non-trivial rep Γ is not faithful, then G has at least one non-trivial normal subgroup H, such that Γ induces a faithful representation of the quotient group G/H. (in the above sense)

Definition: (unitary representation)
A representation $\Gamma : G \to \mathrm{GL}(V)$ is called unitary, if $\Gamma(g)$ is unitary $\forall \; g \in G$, i.e. $\langle \Gamma(g)v | \Gamma(g)w \rangle = \langle v|w \rangle \; \forall \; v, w \in V$.

Remarks:

1. If V is finite-dimensional and if we choose an orthonormal basis, then such a representation is in terms of unitary matrices.
2. Unitary representations are important for applications in physics, since it is in terms of them that symmetries are implemented in quantum mechanics (or quantum field theory).
3. For finite groups every (finite dimensional) rep is equivalent to a unitary rep, see next section.

2.2 Equivalent Representations

Definition: (equivalent representations)
We say that two representations $\Gamma : G \to \mathrm{GL}(V)$ and $\tilde{\Gamma} : G \to \mathrm{GL}(W)$ are equivalent, if there exists an invertible linear map $S : V \to W$ such that

$$\Gamma(g) = S^{-1}\,\tilde{\Gamma}(g)\,S \quad \forall\, g \in G\,.$$

Remarks:

1. If the linear map is even unitary, i.e. (writing U instead of S) $U : V \to W$ with $\langle U\phi | U\psi \rangle_W = \langle \phi|\psi \rangle_V$ then we say that the representations are *unitarily equivalent.* For finite-dimensional representations we have $V \cong W \cong \mathbb{C}^{\dim V}$, and by choosing orthonormal bases U becomes a unitary matrix.
2. For finite groups every representation is equivalent to a unitary representation...

Theorem 2. *Let G be a finite group, $\Gamma : G \to \mathrm{GL}(V)$ a representations and $\langle \cdot | \cdot \rangle$ a scalar product on V. Then Γ is equivalent to a unitary representation.*

Proof:

$$(v,w) := \sum_{g\in G} \langle \Gamma(g)v|\Gamma(g)w\rangle \qquad (*)$$

is also a scalar product since

(i) $(v,w) = \sum_{g\in G} \langle \Gamma(g)v|\Gamma(g)w\rangle = \sum_{g\in G} \overline{\langle \Gamma(g)w|\Gamma(g)v\rangle} = \overline{\sum_{g\in G} \langle \Gamma(g)w|\Gamma(g)v\rangle} = \overline{(v,w)}$,

(ii) $(v,\alpha w) = \sum_{g\in G} \langle \Gamma(g)v|\Gamma(g)\alpha w\rangle = \alpha \sum_{g\in G} \langle \Gamma(g)v|\Gamma(g)w\rangle = \alpha(v,w)$,

(iii) $(v,v) = \sum_{g\in G} \underbrace{\langle \Gamma(g)v|\Gamma(g)v\rangle}_{\geq 0} \geq \langle \Gamma(e)v|\Gamma(e)v\rangle = \langle v|v\rangle \geq 0$, $= 0$ only, if $v = 0$.

Let $\{v_j\}$ be an orthonormal basis (ONB) with respect to $\langle\cdot|\cdot\rangle$ and $\{w_j\}$ an ONB with respect to $(\cdot,\cdot)$. Then there exists an invertible map $S : V \to V$ with $Sw_j = v_j$ (change of basis). Hence

$$(v,w) = \langle Sv|Sw\rangle\,, \qquad (+)$$

since with $v = \sum_j \alpha_j w_j$ and $w = \sum_j \beta_j w_j$ we see that

$$\langle Sv|Sw\rangle = \langle S\sum_j \alpha_j w_j | S\sum_k \beta_k w_k\rangle = \sum_{j,k} \overline{\alpha_j}\beta_k \underbrace{\langle v_j|v_k\rangle}_{=\delta_{jk}=(w_j,w_k)} = (\sum_j \alpha_j w_j, \sum_k \beta_k w_k) = (v,w)\,.$$

Now $\tilde{\Gamma}$ with

$$\tilde{\Gamma}(g) := S\Gamma(g)S^{-1}$$

is equivalent to Γ and unitary, since

$$\begin{aligned}
\langle \tilde{\Gamma}(g)v|\tilde{\Gamma}(g)w\rangle &= \langle S\Gamma(g)S^{-1}v|S\Gamma(g)S^{-1}w\rangle \\
&\underset{(+,*)}{=} \sum_{g'\in G} \langle \underbrace{\Gamma(g')\Gamma(g)}_{\Gamma(g'g)} S^{-1}v|\Gamma(g')\Gamma(g)S^{-1}w\rangle\,, \qquad g'g =: h \\
&= \sum_{h\in G} \langle \Gamma(h)S^{-1}v|\Gamma(h)S^{-1}w\rangle \qquad \text{(rearrangement lemma)} \\
&\underset{(*)}{=} (S^{-1}v, S^{-1}w) \\
&\underset{(+)}{=} \langle v|w\rangle
\end{aligned}$$

□

Remark: Finiteness of G was necessary in order to be able to write $\sum_{g\in G}$. Later we will see, that for some infinite groups (namely compact groups, like e.g. SO(n) or U(n)) we can replace the sum by a suitable integral. The theorem then still holds for continuous representations.

2.3 Beispiele und Invariante Unterräume

— section skipped in WS 19/20 —

Wir führen einige wichtige Konzepte zusammen mit einigen Sprechweisen aus der physikalischen Literatur anhand eines einfachen Beispiels ein.

- Betrachte wieder $\{I, P\} \cong \mathbb{Z}_2$,

  $$I : \mathbb{R}^d \ni \vec{x} \mapsto \vec{x}\,, \qquad P : \mathbb{R}^d \ni \vec{x} \mapsto -\vec{x}$$

  sowie $\{O_I, O_P\} \cong \mathbb{Z}_2$ (vgl. Beispiel 3b aus Abschnitt 1.3).[10]

  $$(O_I f)(\vec{x}) = f(\vec{x})\,, \quad (O_P f)(\vec{x}) = f(-\vec{x})\,.$$

  Wähle eine Funktion f_1 ohne spezielle Symmetrieeigenschaften unter $\{O_I, O_P\}$ und definiere

  $$f_2(\vec{x}) := (O_P f_1)(\vec{x}) = f_1(-\vec{x})\,.$$

  Weiter sei

  $$\mathcal{S} := \mathrm{span}(f_1, f_2)\,,$$

  $\dim \mathcal{S} = 2$ (Das war mit "ohne spezielle Symmetrieeigenschaften" gemeint.)

- Man sagt $\mathcal{S}$ ist *invariant* unter $\{O_I, O_P\}$, d.h.

  $$f \in \mathcal{S} \quad \Rightarrow \quad O_I f, O_P f \in \mathcal{S}\,.$$

  Klar, da

  $$O_P f = O_P(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 O_P f_1 + \alpha_2 O_P f_2 = \alpha_2 f_1 + \alpha_1 f_2 \in \mathcal{S}\,.$$

  Dies definiert eine 2-dimensionale Darstellung von $\mathbb{Z}_2$ (oder irgendeiner zu $\mathbb{Z}_2$ isomorphen Gruppe) auf $\mathcal{S}$. In der Basis $\{f_1, f_2\}$ gilt

  $$\Gamma^{③}(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad \Gamma^{③}(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

- Definiere nun eine neue Basis,

  $$\bar{f}_1 := f_1 + f_2\,, \qquad \bar{f}_2 := f_1 - f_2\,, \qquad \mathcal{S} = \mathrm{span}(\bar{f}_1, \bar{f}_2)\,.$$

  $$\Rightarrow \qquad O_P \bar{f}_1 = \bar{f}_1 \quad \text{(gerade)}\,, \qquad O_P \bar{f}_2 = -\bar{f}_2 \quad \text{(ungerade)}\,.$$

[10] $\{O_I, O_P\}$ ist auch eine Darstellung von $\mathbb{Z}_2$ auf einem geeigneten Funktionen-Raum – jetzt wollen wir aber auf etwas anderes hinaus…

Man sagt $\bar{f}_1$ und $\bar{f}_2$ haben feste Parität.
Darstellung von $\mathbb{Z}_2$ auf $\mathcal{S}$ in der neuen Basis:

$$\Gamma^{④}(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^{④}(P) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Gamma^{④}$ ist äquivalent zu $\Gamma^{③}$, sogar unitär äquivalent, denn

$$\Gamma^{④} = U^\dagger \Gamma^{③} U \quad \text{mit} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(Hier klar, denn gerade durch diesen Basiswechsel hatten wir $\Gamma^{④}$ ja erhalten – in anderen Fällen weiß man das aber vielleicht gerade nicht!)

- $\mathcal{S}$ hat jedoch noch kleinere *invariante Unterräume*, es gilt nämlich

$$\mathcal{S} = \bar{\mathcal{S}}_1 \oplus \bar{\mathcal{S}}_2\,, \qquad \text{(direkte Summe)}$$

wobei die $\bar{\mathcal{S}}_j := \text{span}(\bar{f}_j)$ einzeln invariant unter $\{O_I, O_P\}$ sind,

$$\begin{aligned} O_P(\alpha \bar{f}_1) &= \alpha \bar{f}_1 \in \bar{\mathcal{S}}_1 \\ O_P(\alpha \bar{f}_2) &= -\alpha \bar{f}_2 \in \bar{\mathcal{S}}_2 \end{aligned}$$

Man sagt $\mathcal{S}$ ist *reduzibel* (bzgl. $\{O_I, O_P\}$).
$\bar{\mathcal{S}}_1$ und $\bar{\mathcal{S}}_2$ sind *irreduzibel*, d.h. sie können nicht in kleinere invariante Räume zerlegt werden (hier weil sie 1-dimensional sind).

- Auf den invarianten Unterräumen sind jeweils eindimensionale Darstellungen definiert:

$$\begin{aligned} &\Gamma^{①}(I) = 1\,, \qquad \Gamma^{①}(P) = 1\,, \qquad \text{auf } \bar{\mathcal{S}}_1 \qquad \text{und} \\ &\Gamma^{②}(I) = 1\,, \qquad \Gamma^{②}(P) = -1, \qquad \text{auf } \bar{\mathcal{S}}_2\,. \end{aligned}$$

Jede Funktion mit gerader (ungerader) Parität transformiert sich unter $\{O_I, O_P\}$ in der Darstellung $\Gamma^{①}$ ($\Gamma^{②}$).

- Wie $\mathcal{S}$ (s.o.) heißt nun auch die Darstellung $\Gamma^{③}$ *reduzibel*[11] und man schreibt

$$\Gamma^{③} = \Gamma^{①} \oplus \Gamma^{②}\,.$$

- **Weiteres Beispiel:** Betrachte

$$\begin{aligned} &h_1(\vec{x}) := x^2 + y + z\,, \quad h_2(\vec{x}) := (O_P h_1)(\vec{x}) = x^2 - y - z\,, \quad \mathcal{S}_h := \text{span}(h_1, h_2)\,, \\ &g_1(\vec{x}) := e^{-xyz}\,, \qquad g_2(\vec{x}) := (O_P g_1)(\vec{x}) = e^{xyz}\,, \qquad \mathcal{S}_g := \text{span}(g_1, g_2)\,. \end{aligned}$$

[11] wird später noch richtig definiert

Das Tensor-Produkt $\mathcal{S}_h \otimes \mathcal{S}_g$ wird durch die vier Produkte $h_1g_1, h_1g_2, h_2g_1, h_2g_2$ aufgespannt und ist invariant unter $\{O_I, O_P\}$, denn $f \in \mathcal{S}_h \otimes \mathcal{S}_g \Rightarrow$

$$\begin{aligned} O_P f &= O_P(ah_1g_1 + bh_1g_2 + ch_2g_1 + dh_2g_2) \\ &= dh_1g_1 + ch_1g_2 + bh_2g_1 + ah_2g_2 \quad \in \mathcal{S}_h \otimes \mathcal{S}_g \end{aligned}$$

Dies definiert eine 4-dimensionale Darstellung von $\mathbb{Z}_2$ auf $\mathcal{S}_h \otimes \mathcal{S}_g$:

$$\Gamma^{⑤}(I) = \mathbb{1}\,, \qquad \Gamma^{⑤}(P) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Invariante Unterräume:
  h_1g_1 und $h_2g_2 = O_P(h_1g_1)$ spannen einen invarianten Unterraum S^ω auf, analog $\mathcal{S}^\delta := \mathrm{span}(h_1g_2, h_2g_1)$. Offensichtlich:

  $$\mathcal{S}_h \otimes \mathcal{S}_g = \mathcal{S}^\omega \oplus \mathcal{S}^\delta$$

  jeweils mit einer Darstellung äquivalent zu $\Gamma^{③}$. Reduziere $\mathcal{S}^\omega$ und $\mathcal{S}^\delta$ jeweils weiter durch Einführen von Basisfunktionen gerader und ungerader Parität. Für die Darstellungen gilt dann

  $$\Gamma^{⑤} = \Gamma^{③} \otimes \Gamma^{③} = \Gamma^{①} \oplus \Gamma^{①} \oplus \Gamma^{②} \oplus \Gamma^{②}$$

  Man schreibt auch (Dimensionen)

  $$2 \otimes 2 = 1 \oplus 1 \oplus 1 \oplus 1$$

  Sieht etwas lustig aus und ist hier natürlich nicht besonders tiefsinnig – aber wenn wir ähnliche Rechungen später z.B. für Darstellungen von $\mathrm{SU}(n)$ durchführen können, haben wir einiges gelernt...

— end of skipped part —

2.4 Irreducible Representations

This basis way of thinking about $X \otimes Y$ is useful; the abstract definition is useful in showing that the construction is not basis dependent.

Barry Simon

Reminder: (direct sum & tensor product)
Let V and W be vector spaces, $\dim V = n$, $\dim W = m$, with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$, respectively. Then

(i) $\{v_1, \ldots, v_n, w_1, \ldots, w_m\}$ is a basis for the direct sum $V \oplus W$ with $\dim V \oplus W = \dim V + \dim W$ and

(ii) $\{v_j \otimes w_k\}_{j=1,\ldots,n,k=1,\ldots,m}$ is a basis for the tensor product $V \otimes W$ with $\dim V \otimes W = \dim V \cdot \dim W$.

Remarks:

1. For linear maps $A : V \to V$ and $B : W \to W$ we define $A \oplus B$ as the linear map

$$\begin{aligned} A \oplus B \ : \ V \oplus W &\to V \oplus W \\ (v, w) &\mapsto (Av, Bw)\,, \end{aligned}$$

   in matrix notation
$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} Av \\ Bw \end{pmatrix}.$$

2. Given two representations $\Gamma : G \to \mathrm{GL}(V)$ and $\tilde{\Gamma} : G \to \mathrm{GL}(W)$ we can define the representation $\Gamma \oplus \tilde{\Gamma} : G \to \mathrm{GL}(V \oplus W)$, by $(\Gamma \oplus \tilde{\Gamma})(g) = \Gamma(g) \oplus \tilde{\Gamma}(g)$. *(direct sum of representations)*

Product representations $\Gamma \otimes \tilde{\Gamma}$ will be defined similarly later.

In the following we ask ourselves whether a given representation is a direct sum of "smaller" representations. . .

Definition: (invariant subspace)
Let $\Gamma : G \to \mathrm{GL}(V)$ be a representation and $U \subseteq V$ a subspace of V. U is called invariant subspace (with respect to Γ), if $\Gamma(g)v \in U \ \forall\, v \in U$ and $\forall\, g \in G$.

Remark: Every carrier space has two trivial invariant subspaces, namely V and $\{0\}$. All other invariant subspace (if there are any) are called non-trivial.

Definition: (irreducible representation & complete reducibility)
We call a representation $\Gamma : G \to \mathrm{GL}(V)$

(i) irreducible, if V possesses no non-trivial invariant subspace. Then we also call V irreducible with respect to Γ.

(ii) reducible, if V possesses a non-trivial invariant subspace U.

(iii) completely reducible, if V can be written as a direct sum of irreducible invariant subspaces.

Abbreviation for "irreducible representation": *irrep*

Beispiele:
In Abschnitt 2.3 waren $\Gamma^{③}$, $\Gamma^{④}$ und $\Gamma^{⑤}$ reduzibel, $\Gamma^{①}$ und $\Gamma^{②}$ dagegen irreduzibel.

Theorem 3. *Let $\Gamma : G \to \mathrm{GL}(V)$ be a unitary representation and $U \subseteq V$ an invariant subspace. Then:*

(i) $U^{\perp} = \{v \in V \,:\, \langle u|v\rangle = 0 \quad \forall\, u \in U\}$ *is also invariant,*

(ii) *the restrictions $\Gamma|_U$ and $\Gamma|_{U^\perp}$ define representations Γ^1 and Γ^2, and*

(iii) Γ *ist equivalent to $\Gamma^1 \oplus \Gamma^2$; we simply write $\Gamma = \Gamma^1 \oplus \Gamma^2$.*

Corollary: (Maschke's Theorem)
We can write every (finite-dimensional) unitary representation as a direct sum of irreducible representations.

Combined with Theorem 2 this implies that for finite groups *every* (finite-dimensional) representation is completely reducible.

We can find a basis of V such that in matrix notation

$$\Gamma(g) = \begin{pmatrix} \Gamma^1(g) & & & \mathbf{0} \\ & \Gamma^2(g) & & \\ & & \Gamma^3(g) & \\ \mathbf{0} & & & \ddots \end{pmatrix},$$

where the Γ^j are irreducible ($n_j \times n_j$ blocks with $n_j = \dim \Gamma^j$).

Here an irreducible representation can appear more than once, (relabel)

$$\Gamma = \underbrace{\Gamma^1 \oplus \cdots \oplus \Gamma^1}_{a_1 \text{ times}} \oplus \underbrace{\Gamma^2 \oplus \cdots \oplus \Gamma^2}_{a_2 \text{ times}} \oplus \cdots = \bigoplus_j a_j \Gamma^j\,,$$

i.e. in Γ the irreducible representation Γ^j is contained a_j times.

Beispiele: In Abschnitt 2.3 lag die reduzible Darstellung $\Gamma^{④}$ bereits in reduzierter Form (d.h. blockdiagonal) vor, $\Gamma^{③}$ und $\Gamma^{⑤}$ können durch einen Basiswechsel in diese Form gebracht werden. In $\Gamma^{⑤}$ kamen die Irreps $\Gamma^{①}$ und $\Gamma^{②}$ je zweimal vor.

Proof: Essentially, we have to show (i), then (ii) and (iii) follow immediately.
(i) Let $v \in U^\perp$, $u \in U$ and $g \in G$. Then we have

$$\langle \Gamma(g)v|u\rangle = \langle v|\Gamma(g)^\dagger u\rangle = \langle v|\Gamma(g)^{-1}u\rangle = \langle v|\Gamma(g^{-1})u\rangle = 0\,.$$

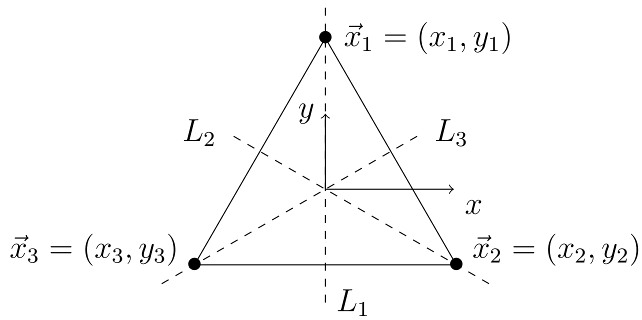
(ii) $\Gamma^1 := \Gamma|_U$, $u \in U \Rightarrow$

$$\Gamma^1(g)\Gamma^1(h)u = \Gamma^1(g)\Gamma(h)u = \Gamma(g)\Gamma(h)u = \Gamma(gh)u = \Gamma^1(gh)u$$

□

2.4.1 Example: O_A operators for the group D_3

- D_3 = symmetry group of an equilateral triangle $\cong S_3$



- group elements:

  e = identity
  C = rotation by 120°, clockwise about the centre $\widehat{=}$ (123)
  $\bar{C}$ = rotation by 120°, anti-clockwise about the centre $\widehat{=}$ (132)
  $\sigma_1, \sigma_2, \sigma_3$ = reflections across $L_1, L_2, L_3 \widehat{=} (23), (13), (12)$

  group table: see exercises

- Now consider invertible linear maps $A : \mathbb{R}^2 \to \mathbb{R}^2, \vec{x} \mapsto A\vec{x}$. (The 6 elements of D_3 are examples for maps of this kind.)

- For each map A define an operator O_A, acting on functions $f : \mathbb{R}^2 \to \mathbb{C}$ (or $\mathbb{R}$) as

$$(O_A f)(\vec{x}) = f(A^{-1}\vec{x}) \,.$$

- The 6 operators O_A, $A \in D_3$, form the group $\bar{D}_3$, isomorphic to D_3, since

$$\begin{aligned}((O_A O_B)f)(\vec{x}) = (O_A(O_B f))(\vec{x}) &= (O_B f)(A^{-1}\vec{x}) = f(B^{-1}A^{-1}\vec{x}) \\ &= f((AB)^{-1}\vec{x}) = (O_{AB} f)(\vec{x}) \,.\end{aligned}$$

- We now let these operators act on some functions, thereby generating representations of $\bar{D}_3 \cong D_3 \cong S_3$.

  First

$$\phi_1(\vec{x}) := \mathrm{e}^{-|\vec{x}-\vec{x}_1|^2} = \mathrm{e}^{-(x-x_1)^2-(y-y_1)^2} \,.$$

  What is $O_C \phi_1$?

$$\begin{aligned}\phi_2(\vec{x}) := (O_C \phi_1)(\vec{x}) &= \phi_1(C^{-1}\vec{x}) \\ &= \exp(-|C^{-1}\vec{x} - \vec{x}_1|^2) \\ &= \exp(-|C^{-1}(\vec{x} - C\vec{x}_1)|^2) \\ &= \exp(-|\vec{x} - C\vec{x}_1|^2) \qquad \text{(rotations conserve lengths)} \\ &= \exp(-|\vec{x} - \vec{x}_2|^2)\end{aligned}$$

Similarly:

$$\phi_3(\vec{x}) := (O_{\bar{C}}\phi_1)(\vec{x}) = \mathrm{e}^{-|\vec{x}-\vec{x}_3|^2}$$

For the reflections we have

$$\begin{aligned}
(O_{\sigma_1}\phi_1)(\vec{x}) &= \phi_1(\sigma_1^{-1}\vec{x}) \\
&= \exp(-|\sigma_1^{-1}\vec{x} - \vec{x}_1|^2) \\
&= \exp(-|\sigma_1^{-1}(\vec{x} - \sigma_1\vec{x}_1)|^2) \\
&= \exp(-|\vec{x} - \sigma_1\vec{x}_1|^2) \qquad \text{(reflections conserve lengths)} \\
&= \exp(-|\vec{x} - \vec{x}_1|^2) \qquad \text{(since } \vec{x}_1 \text{ lies on the } L_1\text{-axis)} \\
&= \phi_1(\vec{x})\,,
\end{aligned}$$

and also

$$\begin{aligned}
(O_{\sigma_2}\phi_1)(\vec{x}) &= \phi_1(\sigma_2^{-1}\vec{x}) = \exp(-|\vec{x} - \sigma_2\vec{x}_1|^2) = \exp(-|\vec{x} - \vec{x}_3|^2) \\
&= \phi_3(\vec{x}) \\
(O_{\sigma_3}\phi_1)(\vec{x}) &= \phi_1(\sigma_3^{-1}\vec{x}) = \exp(-|\vec{x} - \sigma_3\vec{x}_1|^2) = \exp(-|\vec{x} - \vec{x}_2|^2) \\
&= \phi_2(\vec{x})\,.
\end{aligned}$$

Similarly we find out how the Os act on ϕ_2 and ϕ_3,

	ϕ_1	ϕ_2	ϕ_3
O_e	ϕ_1	ϕ_2	ϕ_3
O_C	ϕ_2	ϕ_3	ϕ_1
$O_{\bar{C}}$	ϕ_3	ϕ_1	ϕ_2
O_{σ_1}	ϕ_1	ϕ_3	ϕ_2
O_{σ_2}	ϕ_3	ϕ_2	ϕ_1
O_{σ_3}	ϕ_2	ϕ_1	ϕ_3

,

i.e. $\mathcal{S} := \mathrm{span}(\phi_1, \phi_2, \phi_3)$ is invariant under $\bar{D}_3$, and the functions ϕ_1, ϕ_2, ϕ_3 transform in a three-dimensional representation of the group D_3 ($\cong \bar{D}_3 \cong S_3$), namely

$$\Gamma^1(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma^1(C) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Gamma^1(\bar{C}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Gamma^1(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Gamma^1(\sigma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Gamma^1(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Is this representation reducible?
  Yes, since $\mathcal{S}$ is reducible, ie. there exists a change of basis decomposing $\mathcal{S}$ in smaller

invariant subspaces :

$$\tilde{\phi}_1 = \phi_1 + \phi_2 + \phi_3$$
$$\tilde{\phi}_2 = \sqrt{3}(\phi_2 - \phi_3)$$
$$\tilde{\phi}_3 = 2\phi_1 - \phi_2 - \phi_3$$

(Later we will learn how to find this change of basis.)

- $\tilde{\phi}_1$ is invariant under $\bar{D}_3$, since the operators O_A just permute the terms of the sum, and in particular span($\tilde{\phi}_1$) is invariant and $\tilde{\phi}_1$ transforms in the trivial representation $\Gamma^2(g) = 1 \ \forall g \in D_3$.

- For $\tilde{\phi}_2$ and $\tilde{\phi}_3$ we obtain

	$\tilde{\phi}_2$	$\tilde{\phi}_3$
O_e	$\tilde{\phi}_2$	$\tilde{\phi}_3$
O_C	$-\frac{1}{2}\tilde{\phi}_2 - \frac{\sqrt{3}}{2}\tilde{\phi}_3$	$\frac{\sqrt{3}}{2}\tilde{\phi}_2 - \frac{1}{2}\tilde{\phi}_3$
$O_{\bar{C}}$	$-\frac{1}{2}\tilde{\phi}_2 + \frac{\sqrt{3}}{2}\tilde{\phi}_3$	$-\frac{\sqrt{3}}{2}\tilde{\phi}_2 - \frac{1}{2}\tilde{\phi}_3$
O_{σ_1}	$-\tilde{\phi}_2$	$\tilde{\phi}_3$
O_{σ_2}	$\frac{1}{2}\tilde{\phi}_2 - \frac{\sqrt{3}}{2}\tilde{\phi}_3$	$-\frac{\sqrt{3}}{2}\tilde{\phi}_2 - \frac{1}{2}\tilde{\phi}_3$
O_{σ_3}	$\frac{1}{2}\tilde{\phi}_2 + \frac{\sqrt{3}}{2}\tilde{\phi}_3$	$\frac{\sqrt{3}}{2}\tilde{\phi}_2 - \frac{1}{2}\tilde{\phi}_3$

,

i.e. span($\tilde{\phi}_2, \tilde{\phi}_3$) is invariant, and $\tilde{\phi}_2, \tilde{\phi}_3$ transform in the two-dimensional representation,

$$\Gamma^3(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^3(C) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \Gamma^3(\bar{C}) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$
$$\Gamma^3(\sigma_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma^3(\sigma_2) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \Gamma^3(\sigma_3) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

- Hence, $\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3$ transform under $\bar{D}$ in the representation

$$\Gamma^4(g) = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & \Gamma^3(g) \end{pmatrix} \quad \forall\ g \in D_3\,,$$

i.e. $\Gamma^4 = \Gamma^2 \oplus \Gamma^3$. Moreover, we also write $\Gamma^1 = \Gamma^2 \oplus \Gamma^3$, since Γ^1 is equivalent to Γ^4, (even unitarily equivalent)

$$\Gamma^4(g) = U^\dagger \Gamma^1(g) U \quad \text{with} \quad U = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 0 & 2 \\ \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & -\sqrt{3} & -1 \end{pmatrix} \quad \forall g \in D_3\,.$$

Γ^4 is already given in reduced form, Γ^1 not.

- Remaining **question:** Is the two-dimensional representation Γ^3 reducible?

2.5 Schur's Lemmas and orthogonality of irreducible representations

Theorem 4. (Schur's Lemma 1)
Let G be a group, $\Gamma : G \to \mathrm{GL}(V)$ a finite-dimensional, irreducible representation and $A : V \to V$ a linear map. If A commutes with Γ, i.e. $A\Gamma(g) = \Gamma(g)A \ \forall \ g \in G$, then $A = c\mathbb{1}$ for some $c \in \mathbb{C}$.

Proof:
Let λ be an eigenvalue of A, i.e. $\exists \ v \in V, \ v \neq 0 \ : \ (A-\lambda)v = 0$, then

$$(A-\lambda)\Gamma(g)v = \Gamma(g)(A-\lambda)v = 0 \quad \forall \ g \in G\,,$$

and thus $U := \{v \in V \ : \ (A-\lambda)v = 0\}$ is an invariant subspace. Since $U \neq \{0\}$, and since Γ is irreducible, it follows that $U = V$ and hence $A = \lambda\mathbb{1}$. □

Corollary to Theorem 4
For an abelian group G, every unitary irreducible representation has dimension 1.
Proof: exercises.

Theorem 5. (Schur's Lemma 2)
Let G be a group, $\Gamma : G \to \mathrm{GL}(V)$ and $\tilde{\Gamma} : G \to \mathrm{GL}(W)$ two finite-dimensional, unitary irreducible representations and $A : V \to W$ a linear map. If

$$A\Gamma(g) = \tilde{\Gamma}(g)A \quad \forall \ g \in G\,,$$

then $A = 0$ or Γ and $\tilde{\Gamma}$ are unitarily equivalent.

Proof: Replacing g by g^{-1} and taking the Hermitian conjugate, we also have

$$\Gamma(g)A^\dagger = A^\dagger\tilde{\Gamma}(g) \quad \forall \ g \in G\,.$$

This yields

$$A^\dagger A\Gamma(g) = A^\dagger\tilde{\Gamma}(g)A = \Gamma(g)A^\dagger A \quad \forall \ g \in G\,,$$

With Theorem 4 it follows that $A^\dagger A = c\mathbb{1}$ (with c real), i.e. either $c = 0$ and thus $A = 0$ or $U = \frac{1}{\sqrt{c}}A$ is unitary with $\tilde{\Gamma}(g) = U\Gamma(g)U^\dagger \ \forall \, g \in G$. □

Remark: If the representations are not unitary, but if G is finite, then according to Theorem 2: $\exists \ S$ and T, such that $\Gamma'(G) = S\Gamma(G)S^{-1}$ and $\tilde{\Gamma}'(G) = T\Gamma(G)T^{-1}$ are unitary. For $A' := TAS^{-1}$ we have

$$A'\Gamma'(G) = TAS^{-1}S\Gamma(G)S^{-1} = T\tilde{\Gamma}(G)AS^{-1} = \tilde{\Gamma}'(G)A'\,,$$

ie. either $A' = 0$ and thus $A = 0$ or $\exists \ U$ unitary, such that

$$\begin{aligned} & & \tilde{\Gamma}'(G) &= U\Gamma'(G)U^{-1} \\ \Leftrightarrow & & T\tilde{\Gamma}(G)T^{-1} &= US\Gamma(G)S^{-1}U^{-1} \\ \Leftrightarrow & & \tilde{\Gamma}(G) &= T^{-1}US\Gamma(G)S^{-1}U^{-1}T\,, \end{aligned}$$

i.e. Γ and $\tilde{\Gamma}$ are equivalent.

Theorem 6. *Let G be a finite group and Γ^j, $j = 1, 2, \ldots$, non-equivalent unitary irreducible representations with $\dim \Gamma^j = d_j$. Then the matrix elements obey the orthogonality relation*

$$\frac{1}{|G|} \sum_{g \in G} \overline{(\Gamma^j(g)_{\mu\nu})}\, \Gamma^k(g)_{\mu'\nu'} = \frac{1}{d_j} \delta_{jk} \delta_{\mu\mu'} \delta_{\nu\nu'}$$

$\forall\ \mu, \nu = 1, \ldots, d_j$ *and* $\forall\ \mu', \nu' = 1, \ldots, d_k$.

Proof: Let V_j be the carrier space of Γ^j, and $A : V_j \to V_k$ linear (otherwise arbitrary). Define

$$\tilde{A} := \frac{1}{|G|} \sum_{g \in G} \Gamma^k(g) A \Gamma^j(g)^{-1}\,. \qquad (*)$$

For every $h \in G$ we have

$$\begin{aligned}
\Gamma^k(h)\tilde{A} &= \frac{1}{|G|} \sum_{g \in G} \Gamma^k(h)\Gamma^k(g) A \Gamma^j(g)^{-1} \\
&= \frac{1}{|G|} \sum_{g \in G} \Gamma^k(hg) A \Gamma^j(g)^{-1} \\
&= \frac{1}{|G|} \sum_{g' \in G} \Gamma^k(g') A \Gamma^j(h^{-1}g')^{-1} \\
&= \frac{1}{|G|} \sum_{g' \in G} \Gamma^k(g') A \Gamma^j(g')^{-1}\, \Gamma^j(h^{-1})^{-1} \\
&= \tilde{A}\Gamma^j(h)\,.
\end{aligned}$$

With Schur's lemma (Theorem 5) we conclude that $\tilde{A} = 0$ if $j \neq k$, and else $\tilde{A} = c\mathbb{1}$ with

$$c = \frac{1}{d_j} \operatorname{tr} \tilde{A} = \frac{1}{d_j} \operatorname{tr} A\,,$$

i.e.

$$\tilde{A} = \frac{1}{d_j} \operatorname{tr} A\, \delta_{jk} \mathbb{1}\,. \qquad (+)$$

Now choose $A_{\alpha\beta} = \delta_{\alpha\nu'}\delta_{\beta\nu}$ (i.e. only one element $\neq 0$) $\Rightarrow \operatorname{tr} A = \delta_{\nu\nu'}$. Finally:

$$\begin{aligned}
\tilde{A}_{\mu'\mu} &\underset{(+)}{=} \frac{1}{d_j} \delta_{\nu\nu'} \delta_{jk} \delta_{\mu\mu'} \\
&\underset{(*)}{=} \frac{1}{|G|} \sum_{g \in G} \sum_{\alpha,\beta} \Gamma^k(g)_{\mu'\alpha} A_{\alpha\beta} (\Gamma^j(g)^{-1})_{\beta\mu} \\
&= \frac{1}{|G|} \sum_{g \in G} \Gamma^k(g)_{\mu'\nu'} \underbrace{(\Gamma^j(g)^{-1})_{\nu\mu}}_{= (\Gamma^j(g)^\dagger)_{\nu\mu} = \overline{(\Gamma^j(g)_{\mu\nu})}}
\end{aligned}$$

□

Consequences of Theorem 6

- For fixed j, μ, ν we collect the $|G|$ numbers $\Gamma^j(g)_{\mu\nu}$, $g \in G$, in a vector $v^{(j\mu\nu)} \in \mathbb{C}^{|G|}$.
- For each representation Γ^j there are d_j^2 vectors of this kind (since $\mu, \nu = 1, \ldots, d_j$).
- According to Theorem 6 $v^{(j\mu\nu)} \perp v^{(k\mu'\nu')}$, if $j \neq k$ or $\mu \neq \mu'$ or $\nu \neq \nu'$.
- There are at most $|G|$ mutually orthogonal vectors in $\mathbb{C}^{|G|}$

$$\Rightarrow \qquad \sum_j d_j^2 \leq |G| \,.$$

  In Section 2.7 we will show that actually

$$\sum_j d_j^2 = |G| \,.$$

  The sum is over all non-equivalent irreducible representations, i.e., in particular, a finite group has only finitely many non-equivalent finite-dimensional irreducible representations.

2.6 Characters

Definition: (character)
For a finite-dimensional representation $\Gamma : G \to \mathrm{GL}(V)$ we call $\chi : G \to \mathbb{C}$ with

$$\chi(g) = \mathrm{tr}\,\Gamma(g)$$

the character of the representation.

Remarks:

1. In terms of matrix elements we have

$$\chi(g) = \sum_{\mu=1}^{\dim V} \Gamma(g)_{\mu\mu} \,.$$

2. If Γ and $\tilde{\Gamma}$ are equivalent then

$$\tilde{\chi}(g) = \mathrm{tr}\,\tilde{\Gamma}(g) = \mathrm{tr}(S\Gamma(g)S^{-1}) = \mathrm{tr}(S^{-1}S\Gamma(g)) = \mathrm{tr}\,\Gamma(g) = \chi(g) \,.$$

3. All elements of a conjugacy class have the same character,

$$\begin{aligned}\chi(hgh^{-1}) &= \mathrm{tr}\,\Gamma(hgh^{-1}) = \mathrm{tr}\,\big(\Gamma(h)\Gamma(g)\Gamma(h^{-1})\big) = \mathrm{tr}\,\big(\Gamma(h^{-1})\Gamma(h)\Gamma(g)\big) \\ &= \mathrm{tr}\,\big(\Gamma(h^{-1}h)\Gamma(g)\big) = \mathrm{tr}\,\Gamma(g) = \chi(g) \,.\end{aligned}$$

Corollary to Theorem 6. *Let G be a finite group and Γ^j, $j = 1, 2, \ldots$, non-equivalent, irreducible representations with* $\dim \Gamma^j = d_j$. *Then the characters* $\chi^j = \operatorname{tr} \Gamma^j$ *obey the orthogonality relation*

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi^j(g)}\, \chi^k(g) = \delta_{jk}\,.$$

Proof: W.l.o.g. Γ^j unitary (else similarity transform, cf. Theorem 2). In

$$\frac{1}{|G|} \sum_{g \in G} \overline{(\Gamma^j(g)_{\mu\nu})}\, \Gamma^k(g)_{\mu'\nu'} = \frac{1}{d_j} \delta_{jk} \delta_{\mu\mu'} \delta_{\nu\nu'}$$

choose $\nu = \mu$ and $\nu' = \mu'$, and sum over μ and μ'. □

Remarks:

1. Since the characters depend only on the conjugacy class, we can rewrite the orthogonality relation as

   $$\frac{1}{|G|} \sum_{c} n_c\, \overline{\chi_c^j}\, \chi_c^k = \delta_{jk}\,.$$

   Here c labels the classes and n_c is the number of elements in class c.

2. Let m be the number of different conjugacy classes of G and p the number of non-equivalent irreducible representations.

   For fixed j we collect the m numbers χ_c^j in a vector $v^j \in \mathbb{C}^m$. The p vectors for different j are again mutually orthogonal

   $$\Rightarrow \quad p \leq m\,.$$

   We will see (exercises) that in fact $p = m$, i.e. the number of non-equivalent irreducible representations is equal to the number of conjugacy classes.

   The $m \times m$ matrix with entries χ_c^j, $j, c = 1, \ldots, m$, is called *character table* of the group.

3. For a (in general reducible) representation

   $$\Gamma = \bigoplus_j a_j \Gamma^j\,, \quad \Gamma^j \text{ irreducible,}$$

   we have

   $$\chi(g) = \sum_j a_j \chi^j(g)\,.$$

This implies

$$\frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = \frac{1}{|G|}\sum_{j,k} a_j a_k \underbrace{\sum_{g\in G}\overline{\chi^j(g)}\,\chi^k(g)}_{=|G|\delta_{ik}} = \sum_j a_j^2\,.$$

If Γ is irreducible, then one $a_j = 1$ and all others vanish, and thus

$$\frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 1\,.$$

If Γ is reducible, then at least one $a_j > 1$ or several $a_j \neq 0$, and thus

$$\frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 > 1\,.$$

Hence, we have found an *irreducibility criterion* for a given representation.

Example: Representations of $D_3 \cong S_3$ in Section 2.4.1

- conjugacy classes: $\{e\}$, $\{C,\bar{C}\}$, $\{\sigma_1,\sigma_2,\sigma_3\}$
- For the two-dimensional representation Γ^3 we have

$$\frac{1}{|G|}\left(|\chi^3(e)|^2 + |\chi^3(C)|^2\cdot 2 + |\chi^3(\sigma_1)|^2\cdot 3\right) = \frac{2^2+(-1)^2\cdot 2+0}{6} = 1\,,$$

  i.e. Γ^3 is irreducible.
- We have thus found 2 irreducible representations of S_3:
  The trivial representation, which from now on I want to denote as Γ^1 (it was denoted Γ^2 in Section 2.4.1), with $d_1 = 1$ as well as Γ^3 with $d_3 = 2$. From

$$\sum_j d_j^2 = |G| \qquad \text{(We already know } \leq\text{, in Section 2.7 we will show } =\text{.)}$$

  we conclude that there has to be another irreducible representation with dimension $d_2 = 1$ (and no others); it is given by

$$\Gamma^2(e) = \Gamma^2(C) = \Gamma^2(\bar{C}) = 1\,,$$
$$\Gamma^2(\sigma_1) = \Gamma^2(\sigma_2) = \Gamma^2(\sigma_3) = -1$$

  (sign of the corresponding representation).
- Thus the character table of $D_3 \simeq S_3$ reads:

	$\{e\}$	$\{C,\bar{C}\}$	$\{\sigma_1,\sigma_2,\sigma_3\}$
χ^1	1	1	1
χ^2	1	1	-1
χ^3	2	-1	0

Remark: If we know the characters of all irreducible representations of a group, then we can calculate for any given representation (in general reducible) how many times the different irreducible representations appear in it:

$$\underset{\substack{\uparrow\\ \text{character of reducible rep}}}{\chi(g)} = \sum_j \underset{\substack{\uparrow\\ \text{unknown}}}{a_j} \underset{\substack{\uparrow\\ \text{known}}}{\chi^j(g)}$$

$$\Rightarrow \quad \frac{1}{|G|}\sum_{g\in G}\overline{\chi^k(g)}\,\chi(g) = \frac{1}{|G|}\sum_j a_j \underbrace{\sum_{g\in G}\overline{\chi^k(g)}\,\chi^j(g)}_{=|G|\delta_{jk}} = a_k$$

$$\text{or} \quad a_k = \frac{1}{|G|}\sum_c n_c \overline{\chi_c^k}\,\chi_c$$

We call a_j the multiplicity of Γ^j in Γ.

Example: reducible three-dimensional representation Γ of $D_3 \cong S_3$ (denoted Γ^1 in Section 2.4.1:

$$\chi(e) = 3\,, \qquad \chi(C) = \chi(\bar{C}) = 0\,, \qquad \chi(\sigma_1) = \chi(\sigma_2) = \chi(\sigma_3) = 1\,,$$

$$a_1 = \frac{1}{6}[1\cdot 1\cdot 3 + 2\cdot 1\cdot 0 + 3\cdot 1\cdot 1] = 1\,,$$

$$a_2 = \frac{1}{6}[1\cdot 1\cdot 3 + 2\cdot 1\cdot 0 + 3\cdot(-1)\cdot 1] = 0\,,$$

$$a_3 = \frac{1}{6}[1\cdot 2\cdot 3 + 2\cdot(-1)\cdot 0 + 3\cdot 0\cdot 1] = 1\,,$$

character table
n_c

i.e. $\Gamma = \Gamma^1 \oplus \Gamma^3$ as already determined in Section 2.4.1 (different labelling of irreps).

2.7 The regular representation

Definition: (group algebra)
For a finite group G, $|G| = n$, we define its group algebra $\mathcal{A}(G)$ as the vector space spanned by the group elements, i.e. we take (initially formal) linear combinations[12]

$$\mathcal{A}(G) \ni r = \sum_{j=1}^n r_j g_j\,, \quad r_j \in \mathbb{C}\,,$$

with multiplication rule

$$\left(\sum_{j=1}^n r_j g_j\right)\left(\sum_{k=1}^n q_k g_k\right) = \sum_{j=1}^n\sum_{k=1}^n r_j q_k\, g_j g_k\,.$$

induced by group multiplication.

[12]with obvious addition $\sum_{j=1}^n r_j g_j + \sum_{j=1}^n q_j g_j = \sum_{j=1}^n (r_j + q_j) g_j$; multiplication by scalars similarly

Remarks:

1. Due to $g_j g_k \in G$ the result is in $\mathcal{A}(G)$, i.e. the product is well-defined.
2. A matrix representation, say Γ, of G is also a representation of $\mathcal{A}(G)$, in the sense that by defining $\Gamma(\sum_j r_j g_j) = \sum_j r_j \Gamma(g_j)$ we have $\forall\, q, r \in \mathcal{A}(G)$

$$\begin{aligned} \Gamma(qr) &= \Gamma(q)\Gamma(r) \qquad \text{and} \\ \Gamma(q+r) &= \Gamma(q) + \Gamma(r)\,, \end{aligned}$$

where on the r.h.s. we have matrix multiplication and addition, respectively.
3. $\dim \mathcal{A}(G) = |G|$ (as a vector space)
4. Group multiplication can be written as

$$g_j g_k = \sum_{m=1}^{n} g_m \, (\Delta_j)_{mk}\,,$$

where $(\Delta_j)_{mk}$ encodes the group table: For j and k fixed, $(\Delta_j)_{mk} = 1$ for exactly one value of m and vanishes for all others.
5. The $n \times n$ matrices Δ_j, $j = 1, \ldots, n$, with elements

$$(\Delta_j)_{mk}\,, \qquad m, k = 1, \ldots, n\,,$$

form a representation of G, called the *regular representation*.
(Δ_j is the representation matrix for g_j.)

Proof: Let $g_a, g_b, g_c \in G$ with $g_a g_b = g_c \quad \Rightarrow$

$$\begin{aligned} g_a g_b g_j &= \sum_m g_a g_m \, (\Delta_b)_{mj} = \sum_{k,m} g_k \, (\Delta_a)_{km} \, (\Delta_b)_{mj} \\ g_c g_j &= \sum_k g_k \, (\Delta_c)_{kj} \end{aligned}$$

The l.h.s. are identical, and thus also the r.h.s. Compare coefficients:

$$\begin{aligned} (\Delta_c)_{kj} &= \sum_m (\Delta_a)_{km} (\Delta_b)_{mj} = (\Delta_a \Delta_b)_{kj} \\ \Leftrightarrow \quad \Delta_c &= \Delta_a \Delta_b \end{aligned}$$

□

Theorem 7. *(with the above definitions) The regular representation of G contains all irreducible representations of G, and the multiplicity of the irreducible representation Γ^k is given by its dimension d_k,*

$$\Delta = \bigoplus_{k=1}^{p} d_k \, \Gamma^k \qquad \left(p \;=\; \begin{array}{c} \textit{number of non-equivalent} \\ \textit{irreducible representations} \end{array} \right), \qquad (*)$$

i.e. $\exists\, S$ *regular, such that*

$$S^{-1}\,\Delta_j\,S = \begin{pmatrix} 1 & & & & & & & \\ & \Gamma^2(g_j) & & & & & & \\ & & \ddots & & & & & \\ & & & \Gamma^2(g_j) & & & & \\ & & & & \ddots & & & \\ & & & & & \Gamma^m(g_j) & & \\ & & & & & & \ddots & \\ & & & & & & & \Gamma^m(g_j) \end{pmatrix} .$$

$$\underbrace{\qquad\qquad}_{d_2\ \text{blocks}} \cdots \underbrace{\qquad\qquad}_{d_m\ \text{blocks}}$$

Proof: The characters of the regular representation are

$$\chi^R(g_j) = \sum_k (\Delta_j)_{kk}\,.$$

For the identity we have (obviously!)

$$eg_j = \sum_{m=1}^{n} g_m(\Delta_e)_{mj} \qquad \Rightarrow \qquad (\Delta_e)_{mj} = \delta_{mj} \qquad \Rightarrow \qquad \chi^R(e) = n\,.$$

For $g_k \neq e$ gilt

$$g_k g_j = \sum_{m=1}^{n} g_m(\Delta_k)_{mj} \neq g_j \qquad \Rightarrow \qquad (\Delta_k)_{jj} = 0 \qquad \Rightarrow \qquad \chi^R(g_k) = 0\,.$$

With the formula from Section 2.6 we find
(a_k: multiplicity of the k^{th} irreducible representation)

$$a_k = \frac{1}{n}\sum_{j=1}^{n} \overline{\chi^k(g_j)}\,\chi^R(g_j) = \frac{1}{n}\,\overline{\chi^k(e)}\,n = d_k$$

□

Corollary to Theorem 7. *We have*

$$\sum_k d_k^2 = n\,.$$

Here d_k is the dimension of the k^{th} irreducible representation and $n = |G|$.

Remark: In Section 2.5 we only showed $\leq$.

Proof: In $(*)$ choose $g_j = e$,

$$\Delta_e = \bigoplus_k d_k \, \Gamma^k(e) \,,$$

and take the trace,

$$\chi^R(e) = \mathrm{tr}\, \Delta_e = n = \sum_k d_k^2 \,.$$

□

2.8 Product representations and Clebsch-Gordan coefficients

In physics applications one often considers vector spaces that are tensor products, where each factor carries a representation of the same group.

Example: Coupling of angular momenta, e.g. orbital angular momentum and spin of an electron, or spins of several particles – each factor carries a representation of SU(2).

Let U and V be vector spaces with bases $\{u_i\}$ and $\{v_j\}$, respectively, and let $W = U \otimes V$ with basis $\{w_k\}$, where $w_k = u_i \otimes v_j$ (cf. Section 2.4). Further let $A : U \to U$ and $B : V \to V$ be linear maps. Then $D := A \otimes B$ is the linear map $W \to W$ with

$$Dw_k = Au_i \otimes Bv_j \,, \quad \text{where } k = (i,j) \,,$$

and extended by linearity to arbitrary $w \in W$, i.e. for $w = \sum_k \alpha_k w_k$ we have

$$Dw = \sum_{i,j} \alpha_{ij} \, Au_i \otimes Bv_j \,.$$

In matrix components:

$$Au_i = \sum_{i'} u_{i'} A_{i'i} \,, \qquad Bv_j = \sum_{j'} v_{j'} B_{j'j} \quad \text{and}$$

$$Dw_k = \sum_{k'} w_{k'} D_{k'k} = \sum_{i'j'} (u_{i'} \otimes v_j) A_{i'i} \, B_{j'j} \,,$$

i.e. $D_{k'k} \equiv D_{i'j'ij} = A_{i'i} \, B_{j'j}$. If everything is finite-dimensional then

$$\mathrm{tr}\, D = \sum_k D_{kk} = \sum_{i,j} A_{ii} B_{jj} = \mathrm{tr}\, A \cdot \mathrm{tr}\, B = \mathrm{tr}(A \otimes B) \,.$$

Scalar products on U and V induce a scalar product on W by

$$\langle w_k | w_{k'} \rangle := \langle u_i | u_{i'} \rangle_U \, \langle v_j | v_{j'} \rangle_V \,,$$

again extended by (sesqui-)linearity.
If $\{u_i\}$ and $\{v_j\}$ are ONB with respect to $\langle \, | \, \rangle_U$ and $\langle \, | \, \rangle_V$, then $\{w_k\}$ is also orthonormal,

$$\langle w_k | w_{k'} \rangle = \delta_{ii'} \delta_{jj'} = \delta_{kk'} \,.$$

Definition: (product representation)
For representations $\Gamma^\mu : G \to \mathrm{GL}(U)$ and $\Gamma^\nu : G \to \mathrm{GL}(V)$ we define the product representation $\Gamma^{\mu\otimes\nu} : G \to \mathrm{GL}(U \otimes V)$ by

$$\Gamma^{\mu\otimes\nu}(g) = \Gamma^\mu(g) \otimes \Gamma^\nu(g) \quad \forall\, g \in G\,.$$

Remarks:

1. $\Gamma^{\mu\otimes\nu}$ is a representation since

$$\begin{aligned}\Gamma^{\mu\otimes\nu}(gh)w_k &= \Gamma^\mu(gh)u_i \otimes \Gamma^\nu(gh)v_j \\ &= \Gamma^\mu(g)\Gamma^\mu(h)u_i \otimes \Gamma^\nu(g)\Gamma^\nu(h)v_j \\ &= \Gamma^{\mu\otimes\nu}(g)\big(\Gamma^\mu(h)u_i \otimes \Gamma^\nu(h)v_j\big) \\ &= \Gamma^{\mu\otimes\nu}(g)\Gamma^{\mu\otimes\nu}(h)\,\underbrace{(u_i \otimes v_j)}_{=w_k}\,.\end{aligned}$$

2. For the characters we have

$$\chi^{\mu\otimes\nu}(g) = \mathrm{tr}\,\Gamma^{\mu\otimes\nu}(g) = \mathrm{tr}\,\big(\Gamma^\mu(g)\otimes\Gamma^\nu(g)\big) = \mathrm{tr}\,\Gamma^\mu(g)\ \mathrm{tr}\,\Gamma^\nu(g) = \chi^\mu(g)\chi^\nu(g)\,.$$

3. Even for irreducible Γ^μ and Γ^ν the product representation is in general reducible,

$$\Gamma^\mu \otimes \Gamma^\nu = \bigoplus_\lambda a_\lambda \Gamma^\lambda \quad \text{with} \quad \sum_\lambda a_\lambda d_\lambda = d_\mu d_\nu\,,$$

where d_λ is the dimension of Γ^λ. According to Section 2.6 the multiplicities are

$$a_\lambda = \frac{1}{|G|}\sum_c n_c\,\overline{\chi_c^\lambda}\,\chi_c^\mu\chi_c^\nu\,,$$

Example: (cf. Section 1.3)
$\mathbb{Z}_2 = \{e, P\}$, two one-dimensional irreps, character table:

	e	P
$\chi^1 = \Gamma^1$	1	1
$\chi^2 = \Gamma^2$	1	-1

Another rep (reducible)

$$\Gamma^3(e) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,, \qquad \Gamma^3(P) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\,.$$

Define $\Gamma^4 := \Gamma^3 \otimes \Gamma^3 \quad \Rightarrow \quad \chi^4(e) = 2\cdot 2 = 4$, $\chi^4(P) = 0$. Thus,

$$\begin{aligned}a_1 &= \frac{1}{2}(4\cdot 1 + 0\cdot 1) = 2 \quad \text{and} \\ a_2 &= \frac{1}{2}(4\cdot 1 + 0\cdot(-1)) = 2\,,\end{aligned}$$

i.e. $\Gamma^3 \otimes \Gamma^3 = 2\Gamma^1 \oplus 2\Gamma^2$ as one also easily finds explicitly, by diagonalising

$$\Gamma^4(e) = \mathbb{1}_4 \quad \text{and} \quad \Gamma^4(P) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

4. In general we can decompose $W = U \otimes V$ into a direct sum of (under G) invariant irreducible subspaces W^λ_α, with $\dim(W^\lambda_\alpha) = d_\lambda$. The index $\alpha = 1, \ldots, a_\lambda$ distinguishes different subspaces carrying the same irreducible representation, i.e. $\exists\ U$, such that

$$U^{-1}\,\Gamma^{\mu\otimes\nu}\,U = \begin{pmatrix} \boxed{\Gamma^1} & & & & & & \\ & \ddots & & & & & \\ & & \boxed{\Gamma^1} & & & & \\ & & & \ddots & & & \\ & & & & \boxed{\Gamma^\lambda} & & \\ & & & & & \ddots & \\ & & & & & & \boxed{\Gamma^\lambda} \\ \underbrace{\qquad\qquad}_{a_1 \text{ blocks}} & & \cdots & & \underbrace{\qquad\qquad\qquad}_{a_\lambda \text{ blocks}} & & \ddots \end{pmatrix} .$$

Thus U provides the change of basis from the $\{w_k\}$ to some new basis $\{w^\lambda_{\alpha\ell}\}$ in which the representation matrices are block-diagonal. Here $\ell = 1, \ldots, d_\lambda$ numbers the absis vectors of W^λ_α.

By choosing ONBs on both sides U becomes unitary.

Remark: In general U is highly non-unique.

The rest is essentially **notation** – somewhat nasty, but widely used, and sometimes even useful.

With $k = (i, j)$ and in so-called Dirac notation, one writes

$$|w^\lambda_{\alpha\ell}\rangle = \sum_{ij} |w_{ij}\rangle \underbrace{\langle i, j(\mu,\nu)\alpha, \lambda, \ell\rangle}_{\text{Clebsch-Gordan coefficients}} . \qquad (*)$$

The *Clebsch-Gordan coefficients* are matrix elements of U, with

(i, j): row index (old basis),

(α, λ, ℓ): column index (new basis),

(μ, ν): fix. (Tells us which product is decomposed.)

The inverse of $(*)$ is

$$|w_{ij}\rangle = \sum_{\alpha\lambda\ell} |w^\lambda_{\alpha\ell}\rangle\langle\alpha, \lambda, \ell(\mu, \nu)i, j\rangle\,,$$

(this defines $\langle\alpha, \lambda, \ell(\mu, \nu)i, j\rangle$)
and with U unitary we have $\langle\alpha, \lambda, \ell(\mu, \nu)i, j\rangle = \overline{\langle i, j(\mu, \nu)\alpha, \lambda, \ell\rangle}$

- The CG coefficients satisfy "orthonormality and completeness relations"

$$\sum_{\alpha\lambda\ell}\langle i', j'(\mu, \nu)\alpha, \lambda, \ell\rangle\langle\alpha, \lambda, \ell(\mu, \nu)i, j\rangle = \delta_{i'i}\delta_{j'j} \qquad \text{and}$$
$$\sum_{ij}\langle\alpha', \lambda', \ell'(\mu, \nu)i, j\rangle\langle i, j(\mu, \nu)\alpha, \lambda, \ell\rangle = \delta_{\alpha'\alpha}\delta_{\lambda'\lambda}\delta_{\ell'\ell}\,,$$

in matrix notation $U^\dagger U = \mathbb{1} = UU^\dagger$.

- **simplified notation**
  - $|i, j\rangle := |w_{ij}\rangle$ and $|\alpha, \lambda, \ell\rangle := |w^\lambda_{\alpha\ell}\rangle$
  - Einstein summation convention (sum over repeated indices)
  - $\langle i, j(\mu, \nu)\alpha, \lambda, \ell\rangle = \langle i, j|\alpha, \lambda, \ell\rangle$

Then we can write

$$\Gamma^{\mu\otimes\nu}(g)|i, j\rangle = |i', j'\rangle\Gamma^\mu(g)_{i'i}\Gamma^\nu(g)_{j'j} \qquad \text{and}$$
$$\Gamma^{\mu\otimes\nu}(g)|\alpha, \lambda, \ell\rangle = |\alpha, \lambda, \ell'\rangle\Gamma^\lambda(g)_{\ell'\ell}\,,$$

and conclude

$$\begin{aligned}\langle\alpha', \lambda', \ell'|\Gamma^{\mu\otimes\nu}(g)|\alpha, \lambda, \ell\rangle &= \langle\alpha', \lambda', \ell'|\alpha, \lambda, \ell''\rangle\Gamma^\lambda(g)_{\ell''\ell} = \delta_{\alpha'\alpha}\delta_{\lambda'\lambda}\delta_{\ell'\ell''}\Gamma^\lambda(g)_{\ell''\ell}\\ &= \delta_{\alpha'\alpha}\delta_{\lambda'\lambda}\Gamma^\lambda(g)_{\ell'\ell}\\ &\underset{(*)}{=} \langle\alpha', \lambda', \ell'|\Gamma^{\mu\otimes\nu}(g)|i, j\rangle\langle i, j|\alpha, \lambda, \ell\rangle\\ &= \langle\alpha', \lambda', \ell'|i', j'\rangle\Gamma^\mu(g)_{i'i}\Gamma^\nu(g)_{j'j}\langle i, j|\alpha, \lambda, \ell\rangle\,.\end{aligned}$$

(relation between elements of the representation matrices in the old and the new basis)

Example:
In quantum mechanics (the spin degree of freedom of) a spin-$\frac{1}{2}$ particle is described by a vector in $\mathbb{C}^2$. The basis vectors

$$|\uparrow\rangle := \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad |\downarrow\rangle := \begin{pmatrix}0\\1\end{pmatrix}$$

transform in a two-dimensional representation of SU(2), namely $\Gamma^2(g) = g \;\forall\; g \in \mathrm{SU}(2)$. Consider two spin-$\frac{1}{2}$ particles: $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$, spanned by the product basis

$$|\uparrow\uparrow\rangle := |\uparrow\rangle \otimes |\uparrow\rangle\,, \qquad |\uparrow\downarrow\rangle := |\uparrow\rangle \otimes |\downarrow\rangle\,, \qquad |\downarrow\uparrow\rangle := |\downarrow\rangle \otimes |\uparrow\rangle\,, \qquad |\downarrow\downarrow\rangle := |\downarrow\rangle \otimes |\downarrow\rangle\,,$$

transforms in $\Gamma^{2\otimes 2}$. Define a new basis,

$$|0,0\rangle := \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\,, \qquad |1,1\rangle := |\uparrow\uparrow\rangle\,, \qquad |1,0\rangle := \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\,, \qquad |1,-1\rangle := |\downarrow\downarrow\rangle\,.$$

In the exercises we show:

- $|0,0\rangle$ transforms in the spin-0 representation of SU(2) (one-dimensional – trivial representation), and
- $|1,m\rangle$, $m = -1, 0, 1$, transform in the spin-1 representation (three-dimensional) of SU(2).

Clebsch-Gordan coefficients:

	$\|\uparrow\uparrow\rangle$	$\|\uparrow\downarrow\rangle$	$\|\downarrow\uparrow\rangle$	$\|\downarrow\downarrow\rangle$
$\langle 0,0\|$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$\langle 1,1\|$	1	0	0	0
$\langle 1,0\|$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$\langle 1,-1\|$	0	0	0	1

i.e. e.g. $\langle 1,0|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}$.

In general one labels the unitary irreducible representations of SU(2) by their so-called spin quantum number $s \in \frac{1}{2}\mathbb{N}_0$; the correspong representation has dimension $2s+1$.

3 Applications in quantum mechanics

In the following we explore the consequences of the orthogonality relations for irreducible representations (Theorem 6) for degeneracies of quatum mechanical energy levels.

3.1 Expansion in irreducible basis functions and selections rules

In quantum mechanics one considers vector spaces (Hilbert spaces) like $V = L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$, i.e. $\mathbb{C}^n$-valued square-integrable functions in d variables, e.g. $d = 3$ and $n = 2s + 1$ for a particle with spin s, moving in three-dimensional space ($\vec{x} \in \mathbb{R}^3$: position of the particle).
$\psi, \varphi \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^n$, scalar product

$$\langle \psi | \varphi \rangle = \sum_{m=1}^{n} \int_{\mathbb{R}^d} \overline{\psi_m(x)}\, \varphi_m(x)\, \mathrm{d}^d x\,.$$

An operator $A : V \to V$ is called *unitary*, if it leaves scalar products invariant, i.e.

$$\langle A\psi | A\varphi \rangle = \langle \psi | \varphi \rangle \quad \forall\ \psi, \varphi \in V\,.$$

Lemma 8. *Let G be a (finite) group of linear, unitary operators, $A \in G$,*[13] *and let $\psi_1^\nu, \dots, \psi_{d_\nu}^\nu$ be functions that transform in the unitary irreducible representation Γ^ν (with $\dim(\Gamma^\nu) = d_\nu$), i.e.*

$$A\psi_\alpha^\nu = \sum_{\beta=1}^{d_\nu} \psi_\beta^\nu\, \Gamma^\nu(A)_{\beta\alpha}\,. \tag{$*$}$$

Then $\exists\, C_\nu \in \mathbb{C}$ such that

$$\langle \psi_\alpha^\nu | \psi_\beta^\mu \rangle = C_\nu\, \delta_{\nu\mu}\, \delta_{\alpha\beta}\,. \tag{$+$}$$

Remark: We say that the ψ_α^ν have special symmetry properties with respect to G. If $\nu \neq \mu$, then ψ_α^ν and $\psi_{\alpha'}^\mu$ have different symmetry properties. The lemma states that functions with different symmetry properties are orthogonal to each other.

[13] Alternatively, view the operators A as unitary representation of a group G on V.

Proof:

$$
\begin{aligned}
\langle \psi_\alpha^\nu | \psi_\beta^\mu \rangle &= \frac{1}{|G|} \sum_{A\in G} \langle A\psi_\alpha^\nu | A\psi_\beta^\mu \rangle \\
&\underset{(*)}{=} \frac{1}{|G|} \sum_{A\in G} \left\langle \sum_{\gamma=1}^{d_\nu} \psi_\gamma^\nu \, \Gamma^\nu(A)_{\gamma\alpha} \middle| \sum_{\gamma'=1}^{d_\mu} \psi_{\gamma'}^\mu \, \Gamma^\mu(A)_{\gamma'\beta} \right\rangle \\
&= \sum_{\gamma,\gamma'} \underbrace{\frac{1}{|G|} \sum_{A\in G} \overline{(\Gamma^\nu(A)_{\gamma\alpha})} \, \Gamma^\mu(A)_{\gamma'\beta}}_{=\frac{1}{d_\nu}\delta_{\nu\mu}\delta_{\gamma\gamma'}\delta_{\alpha\beta} \text{ (Theorem 6)}} \langle \psi_\gamma^\nu | \psi_{\gamma'}^\mu \rangle \\
&= \delta_{\nu\mu}\delta_{\alpha\beta} \underbrace{\frac{1}{d_\nu} \sum_\gamma \langle \psi_\gamma^\nu | \psi_\gamma^\nu \rangle}_{=C_\nu}
\end{aligned}
$$

□

Remarks:

1. By normalising the ψ_α^ν, $\langle \psi_\alpha^\nu | \psi_\alpha^\nu \rangle = 1$, we get $C_\nu = 1 \; \forall \, \nu$.
2. Now we can express an arbitrary function $\psi \in V$ as linear combination of functions with special symmetry properties (= invariant basis functions) as follows:
   (i) Consider the subspace spanned by the images of ψ under application of all $A \in G$

   $$U = \mathrm{span}(\{A\psi : A \in G\}) \, .$$

   U is invariant under G, and $\psi \in U$.

   (ii) Decompose U into irreducible invariant subspaces (which carry irreducible representations of G), and expand ψ in bases of the invariant subspaces.

   Which irreducible representations, and thus which basis functions, appear in this expansion depends on ψ.
3. Equations like (+) are also called *selection rules.* (Later: A selection rule determines which transitions cannot happen since the transition matrix element vanishes due to symmetries.)

3.2 Invariance of the Hamiltonian and degeneracies

A special role is played by the Hamiltonian $H : V \to V$ (a linear self-adjoint operator) of a quantum mechanical system. In particular, its eigenvalues are the possible energy levels in which we can find the system.

- Let H be the Hamiltonian of a quantum mechanical system and A a unitary operator. If
$$AH = HA\,,$$
then we say A commutes with the Hamiltonian or A leaves H invariant.

- The set of all symmetry operations (realised by unitary operators A_j) which leave H invariant (i.e. $A_jH = HA_j$), forms a group G, the *symmetry group* of H, since
$$\begin{aligned} & A_1H = HA_1\,, \qquad A_2H = HA_2 \\ \Rightarrow \qquad & (A_1A_2)H = A_1A_2H = A_1HA_2 = HA_1A_2 = H(A_1A_2)\,. \end{aligned}$$

- Let $A \in G$ and $|\psi\rangle$ an eigenstate of H with energy E
$$\begin{aligned} & H|\psi\rangle = E|\psi\rangle \\ \Rightarrow \quad & H(A|\psi\rangle) = AH|\psi\rangle = E(A|\psi\rangle) \end{aligned} \qquad (*)$$
i.e. $A|\psi\rangle$ is also eigenstate of H with the same energy E.

- If E is not degenerate then $A|\psi\rangle \propto |\psi\rangle$.

  If E is m-fold degenerate, then $A|\psi\rangle$ is a linear combination of the states $|\psi_1\rangle, \dots, |\psi_m\rangle$ with energy E. (The previous case was just the special case $m = 1$.)

  In any case the space $\mathcal{S} = \text{span}(|\psi_1\rangle, \dots, |\psi_m\rangle)$ is invariant under the symmetry group of H.

  $\Rightarrow$ The degenerate states $|\psi_1\rangle, \dots, |\psi_m\rangle$ transform in a representation of G,
$$A|\psi_j\rangle = \sum_{k=1}^{m} \Gamma(A)_{kj}|\psi_k\rangle\,, \quad A \in G\,. \qquad (+)$$

  In principle this representation can by reducible or irreducible; typically it is irreducible.

  (i) All states transforming in the same irreducible representation of G, must have the same energy:
$$\begin{aligned} & H|\psi_j\rangle = E_j|\psi_j\rangle \\ \underset{(*)}{\Rightarrow} \quad & H(A|\psi_j\rangle) = E_j(A|\psi_j\rangle) \\ \underset{(+)}{\Rightarrow} \quad & H(A|\psi_j\rangle) = \sum_k \Gamma(A)_{kj} \underbrace{H|\psi_k\rangle}_{=E_k|\psi_k\rangle} = \sum_k \Gamma(A)_{kj}E_j|\psi_k\rangle = E_j(A|\psi_j\rangle) \\ & \text{(with } \Gamma \text{ irreducible)} \\ \Rightarrow \quad & E_k\Gamma(A)_{kj} = \Gamma(A)_{kj}E_j \quad \text{(no sums over } j \text{ or } k) \end{aligned}$$

Now define an $m \times m$ diagonal matrix $E = \mathrm{diag}(E_1, \ldots, E_m)$. Then the last equation reads

$$E\,\Gamma(A) = \Gamma(A)E \quad \forall\, A \in G\,,$$

i.e. according to Schur's Lemma (Theorem 4) E is proportional to $\mathbb{1}_m \Rightarrow$ all E_j are identical.

(ii) If Γ is reducible and if $|\psi_j\rangle$ and $|\psi_k\rangle$ transform in different irreducible representations,

$$\begin{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} & \\ & \end{pmatrix} \end{pmatrix} \quad \begin{matrix} \leftarrow j \\ \leftarrow k \end{matrix} \,,$$

then $\Gamma(A)_{ki} = 0$ for all A (Schur's Lemma, Theorem 5) and in general(!) $E_k \neq E_j$, i.e. there is at least no reason why $|\psi_j\rangle$ and $|\psi_k\rangle$ should be degenerate.

(iii) If states transforming in different irreducible representations still have the same energy, we speak about "accidental degeneracy". Possible reasons:

1. "fine-tuning" of or several parameters in H (rather unlikely).
2. We haven't correctly identified the full symmetry group, i.e. we have overlooked some symmetry.

- Conclusions
  - Degenerate states to a given energy typically transform in an irreducible representation of the symmetry group of H. (i.e. they can be classified by irreducible representations).
  - number of degenerate states = dimension of the irreducible representations

Example: Hydrogen atom
First we neglect spin (i.e. in particular no spin-orbit coupling), Hilbert space $L^2(\mathbb{R}^3)$,

$$H = -\frac{\hbar^2}{2m}\Delta - \frac{e^2}{r}\,,$$

where $r = |\vec{x}|$, $\vec{x} \in \mathbb{R}^3$.

- Eigenstates are labelled by so-called quantum numbers

  $n = 1, 2, \ldots$ (principal quantum number),
  $\ell = 0, \ldots, n-1$ (angular/orbital/azimuthal quantum number) and
  $m = -\ell, \ldots, \ell$ (magnetic quantum number),

$$\psi(\vec{x}) = R_{n\ell}(r) Y_{\ell m}(\theta, \phi) \, .$$

- The Hamiltonian for any central force problem, (i.e. H as above, but with $-e^2/r$ replaced by an arbitrary function of r) in 3 dimensions is invariant under O(3). States for fixed n and ℓ transform in a $(2\ell+1)$-dimensional irreducible representation of O(3) (which we will classify later), i.e. the energy does not depend on $m \Rightarrow (2\ell+1)$-fold degeneracy.
- Observation (for hydrogen): The energy also doesn't depend on ℓ ("accidental degeneracy")
$\Rightarrow n^2$-fold degeneracy, since $\sum_{\ell=0}^{n-1}(2\ell+1) = n^2$.
Explanation: The symmetry group is larger than assumed so far. The Hamiltonian of the hydrogen atom is even invariant under O(4) (H commutes also with the Runge-Lenz vector) $\Rightarrow$ energy does not depend on ℓ, and the n^2-fold degeneracy can be understood in terms of the dimensions of the irreducible representations of O(4).

3.3 Perturbation theory and lifting of degeneracies

- typical propblem:

$$H = H_0 + H' \, ,$$

with H_0 "integrable" and H' "small perturbation"

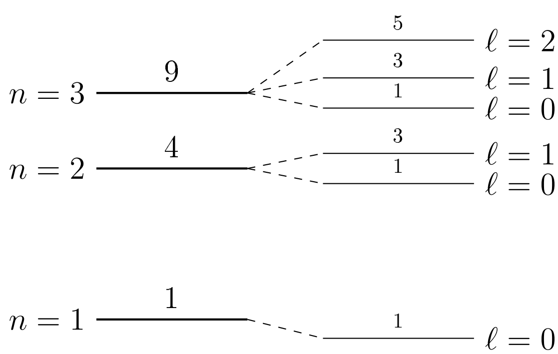
- Let G be the symmetry group of H_0. Two possibilities:
  1. H' is also invariant under G.
  2. H' is only invariant under a subgroup $B \subset G$.
- In case 1 the perturbation H' does not lead to a splitting of levels (it does not lift the degeneracy of the spectrum of H_0).
- Case 2 leads to a splitting of levels (we – partially – lift degeneracies):
  - The exact eigenstates of H transform in irreducible representations of B.
  - The degenerate eigenstates of H_0 transform in irreducible representations of G.
  - For the latter representation, the matrices corresponding to the elements of B, form a representation, say Γ, of B, in general reducible, i.e.

$$\Gamma = \bigoplus_{j=1}^{r} a_j \Gamma^j \qquad \text{with} \qquad \dim(\Gamma^j) = d_j \, .$$

– States transforming in an irreducible representation of B, are still degenerate. States transforming in different irreducible representations of B, in general have different energies, i.e. (some of the) so-far degenerate levels split:
$\Rightarrow \sum_j a_j$ new energy levels
  a_1 of these each d_1-fold degenerate,
  a_2 of these each d_2-fold degenerate, etc.

Examples:

1. Hydrogen atom as in Section 3.2
   Adding a small radially symmetric potential $V(r)$ (but not $\frac{1}{r}$) breaks the O(4)-symmetry to O(3) and each energy level splits into n levels with different ℓ.



   Each new level is still $(2\ell+1)$-fold degenerate, since H' is still invariant under O(3).

2. Fine structure of hydrogen
   - Take electron spin into account: instead of $L^2(\mathbb{R}^3)$ now consider $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.
   - Intermediate step: Consider the same Hamiltonian as before (more precisely $H \to H \otimes \mathbb{1}_2$). States which so far transformed in the representation $\Gamma^{2\ell+1}$ of O(3), now transform[14] in $\Gamma^{2\ell+1} \otimes \Gamma^2$ but energies are unchanged, only the degeneracy is doubled.
   - Now add the perturbation H', containing i.a. spin-dependent terms (spin-orbit coupling), but still invariant under O(3). With

$$\Gamma^{2\ell+1} \otimes \Gamma^2 = \Gamma^{2\ell} \oplus \Gamma^{2\ell+2}$$

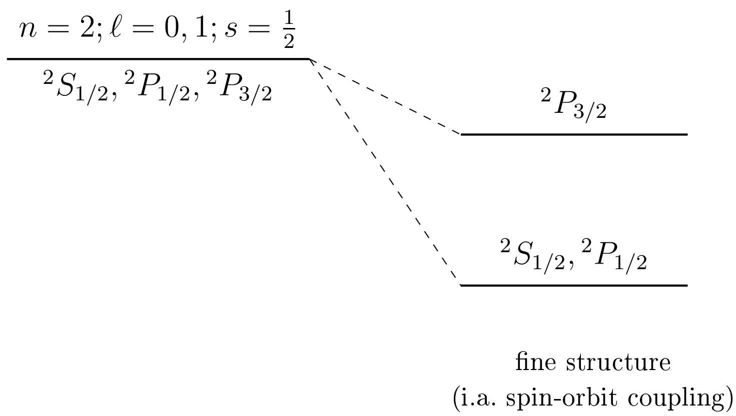
     we obtain states transforming in one of the two irreducible representations. One calls $j = \ell \pm \frac{1}{2}$ the total angular momentum quantum number,

$$2j + 1 = 2(\ell \pm \tfrac{1}{2}) + 1 = \begin{cases} 2\ell + 2 \\ 2\ell \end{cases} .$$

[14] I'm rather sketchy here. Before, we spoke about irreps of SU(2) when discussing spin. Here we first spoke about an O(3)-symmetry. Later we will see that there is an intimate relation between SU(2) and SO(3) (and their irreps) – let's just say by slightly adjusting the perspective it's legitimate to think of $\Gamma^{2\ell+1}$ and Γ^2 as irreps of the same group.

Example: $n = 2$, $\ell = 0, 1$:

$$\underbrace{\Gamma^1 \otimes \Gamma^2}_{\text{s-Orbital, } \ell=0} \oplus \underbrace{\Gamma^3 \otimes \Gamma^2}_{\text{p-Orbital, } \ell=1} = \underbrace{\Gamma^2 \oplus \Gamma^2}_{\substack{\text{still accidentally degenerate,} \\ \text{symmetry group still} \\ \text{larger than O(3)}}} \oplus \Gamma^4$$



4 Expansion into irreducible basis vectors

4.1 Projection operators onto irreducible bases

We take up Remark 2 after Lemma 8: Let U be a representation (e.g. by unitary operators) on V and let $e_1^\nu, \dots, e_{d_\nu}^\nu \in V$ be functions/vectors that transform in the unitary irreducible representation Γ^ν (with $\dim(\Gamma^\nu) = d_\nu$). According to Remark 2 after Lemma 8 we can expand every $\psi \in V$ into such basis vectors, i.e.

$$\psi = \sum_\mu \sum_{\beta=1}^{d_\mu} c_\beta^\mu \, e_\beta^\mu \,,$$

with expansion coefficients $c_\beta^\mu \in \mathbb{C}$. We thus have

$$U(g)\psi = \sum_\mu \sum_{\alpha,\beta} c_\beta^\mu \, e_\alpha^\mu \, \Gamma^\mu(g)_{\alpha\beta} \,,$$

and with Theorem 6 it follows that

$$\frac{d_{\mu'}}{|G|} \sum_{g\in G} \overline{\Gamma^{\mu'}(g)_{\alpha'\beta'}} \, U(g)\psi = \sum_\mu \sum_{\alpha,\beta} c_\beta^\mu \, e_\alpha^\mu \underbrace{\frac{d_{\mu'}}{|G|} \sum_{g\in G} \overline{\Gamma^{\mu'}(g)_{\alpha'\beta'}} \, \Gamma^\mu(g)_{\alpha\beta}}_{=\delta_{\mu\mu'}\delta_{\alpha\alpha'}\delta_{\beta\beta'}} = c_{\beta'}^{\mu'} e_{\alpha'}^{\mu'} \,.$$

Fix μ' and β', and consecutively apply

$$\frac{d_{\mu'}}{|G|} \sum_{g\in G} \overline{\Gamma^{\mu'}(g)_{\alpha'\beta'}} \, U(g) \,, \quad \alpha' = 1, \dots, d_{\mu'} \,,$$

to ψ: Either the result is always zero (if $c_{\beta'}^{\mu'} = 0$) or we obtain $d_{\mu'}$ basis vectors, which transform in $\Gamma^{\mu'}$ (if $c_{\beta'}^{\mu'} \neq 0$).

This motivates the following definition:

Definition: (generalised projection operators)
Let G be a group, U a representation, Γ^μ an irreducible representation, $\dim \Gamma^\mu = d_\mu$. We call

$$P_{jk}^\mu = \frac{d_\mu}{|G|} \sum_{g\in G} [\Gamma^\mu(g)^{-1}]_{jk} \, U(g)$$

generalised projection operator.

Remark: In the following Γ will always be unitary, i.e.

$$[\Gamma^\mu(g)^{-1}]_{jk} = [\Gamma^\mu(g)^\dagger]_{jk} = \overline{\Gamma^\mu(g)_{kj}} \qquad \text{(cf. above).}$$

Theorem 9. (Properties of P^μ_{jk}) *With above definitions we have:*

(i) *For fixed $\psi \in V$ and for fixed μ and j the d_μ vectors $P^\mu_{jk}\psi$, $k = 1, \ldots, d_\mu$, either all vanish or they transform in the irreducible representation Γ^μ.*
In short: $U(g)P^\mu_{jk} = \sum_\ell P^\mu_{j\ell}\,\Gamma^\nu(g)_{\ell k}$.

(ii) $P^\mu_{ji}P^\nu_{\ell k} = \delta_{\mu\nu}\delta_{jk}P^\mu_{\ell i}$.

(iii) $P^\mu_j := P^\mu_{jj}$ *is a projection operator.*

(iv) $P^\mu := \sum_j P^\mu_j$ *is a projection operator onto the invariant subspace U^μ containing all vectors transforming in the irreducible representation Γ^μ.*
($U^\mu = \bigoplus_{\alpha=1}^{a_\mu} U^\mu_\alpha$, U^μ_α: irreducible invariant subspaces, $\alpha = 1, \ldots, a_\mu$, a_μ: multiplicity of Γ^μ in U)

(v) $\sum_\mu P^\mu = \mathbb{1}$ *if V completely reducible. (here always assumed)*

(vi) $U(g) = \sum_\mu \sum_{j,k} \Gamma^\mu(g)_{kj} P^\mu_{jk}$. *(inversion of definition)*

Proof:

(i) see above

(ii) First: action of generalised projection operators on irreducible basis,

$$P^\mu_{ji}\,e^\nu_k = \frac{d_\mu}{|G|}\sum_{g\in G}\overline{\Gamma^\mu(g)_{ij}}\,U(g)\,e^\nu_k = \sum_\ell \underbrace{\frac{d_\mu}{|G|}\sum_{g\in G}\overline{\Gamma^\mu(g)_{ij}}\,\Gamma^\nu(g)_{\ell k}}_{=\delta_{\mu\nu}\delta_{i\ell}\delta_{jk}}\,e^\nu_\ell \qquad (*)$$
$$= \delta_{\mu\nu}\delta_{jk}\,e^\mu_i\,.$$

For $\psi \in V$ arbitrary, we have due to (i): the vectors $\varphi^\nu_k := P^\nu_{\ell k}\psi$ transform in Γ^ν

$$\Rightarrow \quad P^\mu_{ji}P^\nu_{\ell k}\psi = P^\mu_{ji}\,\varphi^\nu_k \underset{(*)}{=} \delta_{\mu\nu}\delta_{jk}\,\varphi^\mu_i = \delta_{\mu\nu}\delta_{jk}\,\varphi^\mu_i = \delta_{\mu\nu}\delta_{jk}\,P^\nu_{\ell i}\psi\,.$$

(iii) $P^\mu_j P^\nu_k = P^\mu_{jj}P^\nu_{kk} \underset{\text{(ii)}}{=} \delta_{\mu\nu}\delta_{jk}\,P^\mu_{jj} = \delta_{\mu\nu}\delta_{jk}\,P^\mu_j$.

(iv)
$$P^\mu P^\nu = \sum_{j,k} P^\mu_j P^\nu_k \underset{\text{(iii)}}{=} \sum_{j,k}\delta_{\mu\nu}\delta_{jk}\,P^\mu_j = \delta_{\mu\nu}\sum_j P^\mu_j = \delta_{\mu\nu}\,P^\mu$$

(v) First: action on irreducible basis,

$$\sum_\mu P^\mu\,e^\nu_k = \sum_\mu\sum_j P^\mu_{jj}\,e^\nu_k = \sum_\mu\sum_j \delta_{\mu\nu}\delta_{jk}\,e^\mu_j = e^\nu_k\,;$$

write $\psi \in V$ as linear combination of irreducible basis vectors $\Rightarrow \sum_\mu P^\mu = \mathbb{1}$.

(vi) For $\psi \in V$ arbitrary we have due to (i): The vectors $\varphi_k^\mu := P_{jk}^\mu \psi$ transform in Γ^μ

$$\Rightarrow \quad \sum_\mu \sum_{j,k} \Gamma^\mu(g)_{kj} P_{jk}^\mu \psi = \sum_\mu \sum_{j,k} \Gamma^\mu(g)_{kj} \varphi_k^\mu = \sum_\mu \sum_j U(g) \varphi_j^\mu$$
$$= U(g) \sum_\mu \sum_j P_{jj}^\mu \psi \underset{(v)}{=} U(g) \psi$$

□

Examples:

1. Reduction of $\mathcal{S} = \text{span}(\phi_1, \phi_2, \phi_3)$ from Section 2.4.1 (invariant under $D_3 \cong S_3$)
   - S_3 has two 1-dimensional and one 2-dimensional irreducible representation $(\Gamma^1, \Gamma^2, \Gamma^3)$.
   - The generalised projection operators are

$$P_{11}^1 = \frac{1}{6}\left(O_I + O_C + O_{\bar{C}} + O_{\sigma_1} + O_{\sigma_2} + O_{\sigma_3}\right) ,$$
$$P_{11}^2 = \frac{1}{6}\left(O_I + O_C + O_{\bar{C}} - O_{\sigma_1} - O_{\sigma_2} - O_{\sigma_3}\right) ,$$
$$P_{11}^3 = \frac{1}{3}\left(O_I - \frac{1}{2}O_C - \frac{1}{2}O_{\bar{C}} - O_{\sigma_1} + \frac{1}{2}O_{\sigma_2} + \frac{1}{2}O_{\sigma_3}\right) ,$$
$$P_{12}^3 = \frac{1}{3}\left(-\frac{\sqrt{3}}{2}O_C + \frac{\sqrt{3}}{2}O_{\bar{C}} - \frac{\sqrt{3}}{2}O_{\sigma_2} + \frac{\sqrt{3}}{2}O_{\sigma_3}\right) ,$$
$$P_{21}^3 = \frac{1}{3}\left(\frac{\sqrt{3}}{2}O_C - \frac{\sqrt{3}}{2}O_{\bar{C}} - \frac{\sqrt{3}}{2}O_{\sigma_2} + \frac{\sqrt{3}}{2}O_{\sigma_3}\right) \quad \text{and}$$
$$P_{22}^3 = \frac{1}{3}\left(O_I - \frac{1}{2}O_C - \frac{1}{2}O_{\bar{C}} + O_{\sigma_1} - \frac{1}{2}O_{\sigma_2} - \frac{1}{2}O_{\sigma_3}\right) .$$

   - Applied to a vector in $\mathcal{S}$, e.g. ϕ_1 (see Section 2.4.1 for the action of the O_A-operators on ϕ_1):
     - $\mu = 1$:

$$P_{11}^1 \phi_1 = \frac{1}{6}\left(\phi_1 + \phi_2 + \phi_3 + \phi_1 + \phi_3 + \phi_2\right) = \frac{1}{3}\left(\phi_1 + \phi_2 + \phi_3\right) ,$$

       invariant under D_3 and transforms in the trivial representation Γ^1.
     - $\mu = 2$:

$$P_{11}^2 \phi_1 = \frac{1}{6}\left(\phi_1 + \phi_2 + \phi_3 - \phi_1 - \phi_3 - \phi_2\right) = 0 ,$$

       had to be zero, since Γ^2 is not contained in the 3-dimensional representation acting on $\mathcal{S}$.

– $\mu = 3$: first $j = 1$,

$$P^3_{11}\phi_1 = \frac{1}{3}\left(\phi_1 - \frac{1}{2}\phi_2 - \frac{1}{2}\phi_3 - \phi_1 + \frac{1}{2}\phi_3 + \frac{1}{2}\phi_2\right) = 0\,,$$

$$P^3_{12}\phi_1 = \frac{\sqrt{3}}{6}\left(-\phi_2 + \phi_3 - \phi_3 + \phi_2\right) = 0 \quad \text{(if one vanishes, then also the other one)}$$

now $j = 2$,

$$P^3_{21}\phi_1 = \frac{\sqrt{3}}{6}\left(\phi_2 - \phi_3 - \phi_3 + \phi_2\right) \propto \phi_2 - \phi_3\,,$$
$$P^3_{22}\phi_1 = \frac{1}{3}\left(\phi_1 - \frac{1}{2}\phi_2 - \frac{1}{2}\phi_3 + \phi_1 - \frac{1}{2}\phi_3 - \frac{1}{2}\phi_2\right) \propto 2\phi_1 - \phi_2 - \phi_3\,.$$

The last two functions transform in Γ^3.

This is the change of basis from Section 2.4.1.

2. Reducing a product representation
   - Let $\Gamma^{\mu\otimes\nu}$ be a product representation of G on $V_\mu \otimes V_\nu$, in general $\Gamma^{\mu\otimes\nu} = \bigoplus_\lambda a_\lambda \Gamma^\lambda$. How do we find the irreducible invariant subspaces of $V_\mu \otimes V_\nu$?
   - Start with a product basis $|k, \ell\rangle = |e^\mu_k\rangle \otimes |e^\nu_\ell\rangle$ and apply the generalised projection operators P^λ_{ji}.
   - For fixed λ, j, k, ℓ the d_λ vectors

     $$P^\lambda_{ji}|k, \ell\rangle\,, \qquad i = 1, \ldots, d_\lambda\,,$$

     either all vanish or they span an irreducible invariant subspace.
   - By varying λ, j, k, ℓ we can find all irreducible invariant subspaces.
   - Exercises: Reduction of $\Gamma^{3\otimes 3}$, where $\Gamma^3 : S_3 \to \mathrm{GL}(\mathbb{C}^2)$.

Summary:

- Decompose the space V into irreducible invariant subspaces,

  $$V = \bigoplus_{\mu,\alpha} V^\mu_\alpha\,,$$

  where μ labels inequivalent irreps and α numbers copies of irrep μ.
- For the basis $|\alpha, \mu, i\rangle$, $i = 1, \ldots, d_\mu$, of V we have

  $$P^\mu|\alpha, \nu, k\rangle = |\alpha, \mu, k\rangle\delta_{\mu\nu}\,,$$
  $$P^\mu_i|\alpha, \nu, k\rangle = |\alpha, \mu, i\rangle\delta_{\mu\nu}\delta_{ik} \quad \text{and}$$
  $$P^\mu_{ij}|\alpha, \nu, k\rangle = |\alpha, \mu, i\rangle\delta_{\mu\nu}\delta_{jk}\,.$$

4.2 Irreducible operators and the Wigner-Eckart Theorem

Definition: (irreducible operators)
Let G be a group, U a representation and Γ^μ a unitary irreducible representation, $\dim \Gamma^\mu = d_\mu$. A set of linear operators, $\{O_i^\mu : i = 1, \ldots, d_\mu\}$, which transform under G as follows,

$$U(g) O_i^\mu U(g)^{-1} = \sum_{j=1}^{d_\mu} O_j^\mu \Gamma^\mu(g)_{ji} \,,$$

is called a set of irreducible operators corresponding to the representation Γ^μ. (The O_i^μ are also called irreducible tensors or irreducible tensor operators).

Remarks:

1. The definition makes sense, since

$$U(gh) O_i^\mu U(gh)^{-1} = U(g)U(h) O_i^\mu U(h)^{-1} U(g)^{-1} = U(g) \sum_j O_j^\mu \Gamma^\mu(h)_{ji} U(g)^{-1}$$
$$= \sum_{j,k} O_k^\mu \Gamma^\mu(g)_{kj} \Gamma^\mu(h)_{ji} = \sum_k O_k^\mu \Gamma^\mu(gh)_{ki} \,.$$

2. Special case: If Γ^μ is the trivial representation then the operator O^μ (no index i, since $d_\mu = 1$) commutes with $U(g)$ $\forall\, g \in G$, cf. Section 3.2.

3. If O_i^μ, $i = 1, \ldots, d_\mu$, are irreducible operators and $|e_j^\nu\rangle$, $j = 1, \ldots, d_\nu$, irreducible basis vectors, then the vectors $O_i^\mu |e_j^\nu\rangle$ transform in the product representation $\Gamma^{\mu\otimes\nu}$:

$$U(g) O_i^\mu |e_j^\nu\rangle = U(g) O_i^\mu U(g)^{-1} U(g) |e_j^\nu\rangle$$
$$= \sum_{k,\ell} O_k^\mu |e_\ell^\nu\rangle \Gamma^\mu(g)_{ki} \Gamma^\nu(g)_{\ell j} \,.$$

   We can reduce this product representation (cf. Section 2.8) and expand the vectors $O_i^\mu |e_j^\nu\rangle$ in the irreducible basis $\{|w_{\alpha\ell}^\lambda\rangle\}$,

$$O_i^\mu |e_j^\nu\rangle = \sum_{\alpha\lambda\ell} |w_{\alpha\ell}^\lambda\rangle \langle \alpha, \lambda, \ell(\mu,\nu) i, j\rangle \,. \qquad (*)$$

   This leads to the…

Theorem 10. (Wigner-Eckart)
Let O_i^μ be irreducible operators and $|e_j^\nu\rangle$ irreducible vectors, then

$$\langle e_\ell^\lambda | O_i^\mu | e_j^\nu \rangle = \sum_\alpha \langle \alpha, \lambda, \ell(\mu,\nu) i, j\rangle \, \langle \lambda \| O^\mu \| \nu \rangle_\alpha$$

with the so-called reduced matrix element (which isn't a matrix element…)

$$\langle \lambda \| O^\mu \| \nu \rangle_\alpha := \frac{1}{d_\lambda} \sum_k \langle e_k^\lambda | w_{\alpha k}^\lambda \rangle \,.$$

Proof:

$$\langle e^\lambda_\ell | O^\mu_i | e^\nu_j \rangle \underset{(*)}{=} \sum_{\alpha,\rho,m} \langle e^\lambda_\ell | w^\rho_{\alpha m} \rangle \, \langle \alpha, \rho, m(\mu,\nu) i, j \rangle$$

In the proof of Lemma 8 (Section 3.1) we showed that

$$\langle e^\lambda_\ell | w^\rho_{\alpha m} \rangle = \delta_{\rho\lambda} \delta_{m\ell} \, \frac{1}{d_\lambda} \sum_k \langle e^\lambda_k | w^\lambda_{\alpha k} \rangle \, ,$$

and thus

$$\langle e^\lambda_\ell | O^\mu_i | e^\nu_j \rangle = \sum_\alpha \underbrace{\frac{1}{d_\lambda} \sum_k \langle e^\lambda_k | w^\lambda_{\alpha k} \rangle}_{=\langle \lambda \| O^\mu \| \nu \rangle_\alpha} \langle \alpha, \lambda, \ell(\mu,\nu) i, j \rangle \, .$$

□

Remarks:

1. The reduced matrix element does not depend on i, j or ℓ. It seems to also not depend on the operators O, and the reps μ and ν, but the $w^\lambda_{\alpha k}$ depend on O, μ and ν, since

$$\mathrm{span}(\{w^\lambda_{\alpha k}\}) = \mathrm{span}(\{O^\mu_i e^\nu_j\})$$

2. Important in applications, since many matrix elements (ME) on the l.h.s. are determined by few reduced MEs on the r.h.s. The latter contain the complete information about the physics. Everything else (CG coefficients) is representation theory, i.e. is already fixed by the symmetries of the problem.
3. In order to determine the reduced MEs calculate as many (suitable) MEs (l.h.s) as there are reduced MEs. Then the Wigner-Eckart Theorem provides us with a system of linear equations for the reduced MEs.

Example: Time-dependent perturbation theory

- Consider an Atom in the state ψ with energy E_ψ under the influence of the (time-dependent) perturbation O (e.g. electromagnetic wave). The probability for a transition to state φ (with energy E_φ) is proportional to

$$|\langle \varphi | O | \psi \rangle|^2 \, .$$

  Thereby, radiation with frequency $|E_\psi - E_\varphi|/h$ is absorbed or emitted. In experiments one observes the intensity of this radiation, which is proportional to $|\langle \varphi | O | \psi \rangle|^2$.
- The unperturbed system is rotationally invariant: ψ and φ are elements of bases transforming in irreducible representations of SO(3): $\Gamma^{2\ell+1}$, $\Gamma^{2\ell'+1}$.
- The perturbation is also rotationally invariant: O is element of a set of irreducible operators, transforms, e.g., in Γ^3 (angular momentum 1, dipole radiation).
- Hence, consider $\langle \ell', m' | O^3_{m''} | \ell, m \rangle$ (further quantum numbers suppressed), $m = -\ell, \ldots, \ell$, $m' = -\ell', \ldots, \ell'$, $m'' = -1, 0, 1$.

- Later we will see: $\Gamma^{3\otimes(2\ell+1)} = \Gamma^{2\ell-1} \oplus \Gamma^{2\ell+1} \oplus \Gamma^{2\ell+3}$, i.e.
  - transitions only possible if $\ell' - \ell = -1, 0, 1 \rightsquigarrow$ selection rule,
  - no α-sum, only one reduced ME,

$$\langle \ell', m' | O^3_{m''} | \ell, m \rangle = \langle \ell', m' (3, 2\ell+1) m'', m \rangle \, \langle \ell' \| O^3 \| \ell \rangle \, .$$

  For fixed ℓ, ℓ' the relative intensities of the $(2\ell+1)(2\ell'+1)$ theoretically possible transitions are already fixed by the CG coefficients – some vanish $\rightsquigarrow$ selection rule.

(Problem slightly simplified here, cf. Wu-Ki Tung, *Group Theory and Physics*, World Scientific, 1985, Sections 4.3, 8.7 & 11.4.)

4.3 Left ideals and idempotents

The generalised projection operators allow us to decompose reducible reps into sums of irreps. To this end we already have to know the irreps. Remaining question: How to construct the irreps?

Reduce the regular representation (see Section 2.7), as it contains all irreducible representations Γ^μ (with multiplicities $d_\mu = \dim(\Gamma^\mu)$).

Recall:

- Carrier space is the group algebra (or Frobenius-Algebra) $\mathcal{A}(G) = \mathrm{span}(g_1, \ldots, g_n)$, $n = |G|$ (group elements numbered again).
- $\mathcal{A}(G) \ni r = \sum_i r_i g_i$, analogously $q \in \mathcal{A}(G)$:

$$rq = \sum_{i,j} r_i q_j \, g_i g_j = \sum_{i,j,k} r_i \, g_k (\Delta_i)_{kj} q_j \, .$$

Definition: (left ideal)
A subspace $L \subseteq \mathcal{A}(G)$ that is invariant under left multiplication is called left ideal, i.e.

$$r \in L \text{ and } q \in \mathcal{A}(G) \quad \Rightarrow \quad qr \in L \, .$$

A left ideal L is called *minimal* if it does not contain any non-trivial left ideal $K \subset L$.

Remarks:

1. Similarly one defines right ideals and two-sided ideals. (Here we only use left ideals.)
2. L is a left ideal $\Leftrightarrow$ L is an invariant subspace, since
   "$\Rightarrow$" o.k., since $G \subset \mathcal{A}(G)$
   "$\Leftarrow$" with $r \in L$ and $q = \sum_j q_j g_j \in \mathcal{A}(G)$ we have

$$qr = \sum_j q_j \underbrace{g_j r}_{\in L \text{ (inv. subspace)}} \qquad \in L \text{ (linear combination of elements } \in L).$$

3. Similarly: L is minimal left ideal $\Leftrightarrow$ L irreducible invariant subspace

Idea: Find the minimal left ideals and construct the irreps which they carry (by applying the group elements to bases for the left ideals).

In the following we denote by P^μ_α the projection operator onto the minimal left ideal L^μ_α, i.e. $P^\mu_\alpha \mathcal{A}(G) = L^\mu_\alpha$. (As before μ labels the non-equivalent irreps, and $\alpha = 1, \ldots, d_\mu$.)

Properties of P^μ_α:

(i) $P^\mu_\alpha r \in L^\mu_\alpha \; \forall \; r \in \mathcal{A}(G)$

(ii) if $q \in L^\mu_\alpha$ then $P^\mu_\alpha q = q$

(iii) $P^\mu_\alpha P^\nu_\beta = \delta_{\mu\nu}\delta_{\alpha\beta} P^\mu_\alpha$,

and it follows that

(iv) $P^\mu_\alpha q = q P^\mu_\alpha \; \forall \; q \in \mathcal{A}(G)$

Proof: Decompose $r \in \mathcal{A}(G)$ as $r = \sum_{\nu,\beta} r^\nu_\beta$ with $r^\nu_\beta \in L^\nu_\beta$. Then

$$q P^\mu_\alpha r = q P^\mu_\alpha \sum_{\nu,\beta} r^\nu_\beta = q r^\mu_\alpha \qquad \text{and}$$
$$P^\mu_\alpha q r = P^\mu_\alpha q \sum_{\nu,\beta} r^\nu_\beta = P^\mu_\alpha \sum_{\nu,\beta} \underbrace{q r^\nu_\beta}_{\in L^\nu_\beta} = q r^\mu_\alpha \, . \qquad \square$$

Now define $L^\mu := \bigoplus_\alpha L^\mu_\alpha$ and first construct the projection operator P^μ onto L^μ:
For each $q \in \mathcal{A}(G)$ exists a unique decomposition

$$q = \sum_\mu q_\mu \quad \text{with} \quad q_\mu \in L^\mu \, ,$$

in particular for the identity,

$$e = \sum_\mu e_\mu \, , \quad e_\mu \in L^\mu \, .$$

Thus,

$$q = qe = q \sum_\mu e_\mu = \sum_\mu \underbrace{q e_\mu}_{\in L^\mu \text{ (since } e_\mu \in L^\mu)} \quad ,$$

i.e. $q_\mu = q e_\mu$, and we have found:

Lemma 11.
P^μ is given by right multiplication with e_μ, i.e. $P^\mu q = q e_\mu \; \forall \; q \in \mathcal{A}(G)$.

Remarks:

1. P^μ is linear.

2. From
$$\underbrace{e_\mu}_{\in L^\mu} = e_\mu e = e_\mu \sum_\nu e_\nu = \sum_\nu \underbrace{e_\mu e_\nu}_{\in L^\nu}$$
it follows that $e_\mu e_\nu = \delta_{\mu\nu} e_\mu$ – cf. property (iii).

3. With $e = \sum_{\mu,\alpha} e^\mu_\alpha$ this also works for projections to minimal left ideals, defined by
$$P^\mu_\alpha q := q e^\mu_\alpha \,.$$

Definition: (idempotents)
An element $e_\mu \in \mathcal{A}(G)$ that satisfies $e_\mu^2 = e_\mu$ is called (an) idempotent. If $e_\mu^2 = \xi_\mu e_\mu$ for some non-zero $\xi_\mu \in \mathbb{C}$ then we call e_μ essentially idempotent.

Remarks:

1. We say the idempotent e_μ *generates* the left ideal L^μ, i.e.
$$L^\mu = \{q e_\mu \;:\; q \in \mathcal{A}(G)\} \,.$$
2. An idempotent is called *primitive*, if it generates a minimal left ideal. Otherwise it can be written as a sum $e_1 + e_2$ of two non-zero idempotents with $e_1 e_2 = 0 = e_2 e_1$.

Theorem 12.
The idempotent e_μ is primitive. $\quad\Leftrightarrow\quad$ *For every $q \in \mathcal{A}(G)$ $\exists\, \lambda_q \in \mathbb{C}$ s.t. $e_\mu q e_\mu = \lambda_q e_\mu$.*

Proof:
"$\Rightarrow$": Let L be the left ideal generated by e_μ.
For $q \in \mathcal{A}(G)$ define the linear map $Q : \mathcal{A}(G) \to \mathcal{A}(G)$ by
$$Qr = r e_\mu q e_\mu \quad \text{for } r \in \mathcal{A}(G) \,.$$
Then $Qsr = sre_\mu q e_\mu = sQr$ $\forall s, r \in \mathcal{A}(G)$, and in particular $\forall r \in L$ and $\forall s \in G$, i.e. Q commutes with the representation of G carried by L.
If e_μ is primitive, then L is minimal and according to Schur's Lemma (Theorem 4) Q is a multiple of the identity on L. The latter is given by right multiplication with e_μ, i.e. $\exists\; \lambda_q \in \mathbb{C}$: $e_\mu q e_\mu = \lambda_q e_\mu$.

"$\Leftarrow$": Let $e_\mu = e_1 + e_2$ with non-zero idempotents $e_1 e_2 = 0 = e_2 e_1$. Then on the one hand
$$e_\mu e_1 e_\mu = (e_1 + e_2) e_1 (e_1 + e_2) = e_1 \,,$$
and on the other hand $\exists\, \lambda \in \mathbb{C}$ s.t.
$$e_\mu e_1 e_\mu = \lambda e_\mu \,.$$
Thus,
$$\lambda e_\mu = e_1 = e_1^2 = \lambda^2 e_\mu^2 = \lambda^2 e_\mu \qquad \Leftrightarrow \qquad \lambda^2 = \lambda \,,$$
but $\lambda = 0$ ↯ $e_1 \neq 0$ and $\lambda = 1 \Rightarrow e_\mu = e_1 \Rightarrow e_2 = 0$ ↯ $e_2 \neq 0$.

□

Theorem 13.
The left ideals generated by two primitive idempotents, e_1 and e_2, carry equivalent irreducible representations Γ^1 and Γ^2 iff $e_1 q e_2 \neq 0$ for at least one $q \in \mathcal{A}(G)$.

Proof:

"$\Leftarrow$": Let $e_1 q e_2 = s \neq 0$ for one $q \in \mathcal{A}(G)$.
Define the linear map $S : \mathcal{A}(G) \to \mathcal{A}(G)$ by $Sr = rs$.
Apparently, $S : L^1 \to L^2$, and since $Se_1 = s \neq 0$ we have $S|_{L^1} \neq 0$.
It follows that $Srp = rps = rSp \; \forall r, p \in \mathcal{A}(G)$, and in particular $\forall r \in G$ and $\forall p \in L^1$, i.e. $S\Gamma^1(r) = \Gamma^2(r)S$. Hence, according to Schur's Lemma (Theorem 5) Γ^1 and Γ^2 are equivalent.

"$\Rightarrow$": If Γ^1 and Γ^2 are equivalent, then there exists a non-trivial linear map $S : L^1 \to L^2$ with $S\Gamma^1(r) = \Gamma^2(r)S \; \forall r \in G$, i.e. $Srp = rSp \; \forall r \in G$ and $\forall p \in L^1$;
by linearity this is also true $\forall r \in \mathcal{A}(G)$.
Define $s := Se_1 \in L^2 \Rightarrow s = se_2$.
Then $s = Se_1 = Se_1e_1 = e_1Se_1 = e_1 s = e_1 s e_2$.

□

Remark:
The primitive idempotent

$$e_1 = \frac{1}{|G|} \sum_{i=1}^{|G|} g_i$$

generates the one-dimensional left ideal L^1, which carries the trivial representation.

Proof: $L^1 = \{re_1 \, : \, r \in \mathcal{A}(G)\}$. With

$$\begin{aligned} re_1 &= \Big(\sum_j r_j g_j\Big)\Big(\frac{1}{|G|}\sum_i g_i\Big) = \sum_j r_j \frac{1}{|G|} \sum_i g_j g_i \\ &= \sum_j r_j \frac{1}{|G|} \sum_k g_k \qquad \text{(rearrangement lemma)} \\ &= c\, e_1 \, , \qquad \text{where} \quad c = \sum\nolimits_j r_j \, , \end{aligned}$$

we find $L^1 = \mathrm{span}(e_1)$, $\dim L^1 = 1$, and thus minimal. Moreover,

$$g \cdot c\, e_1 = \frac{c}{|G|} \sum_i g g_i = \frac{c}{n} \sum_k g_k = c\, e_1$$

i.e. L^1 carries the trivial representation. □

Summary:

- The group algebra $\mathcal{A}(G)$ can be decomposed into left ideals L^μ (μ labels the non-equivalent irreps of the group).
- The L^μ are generated by right multiplication with idempotents e_μ, where

$$e_\mu e_\nu = \delta_{\mu\nu} e_\mu \quad \text{and} \quad \sum_\mu e_\mu = e\,.$$

- Each L^μ can be decomposed into d_μ minimal left ideals L^μ_α, $\alpha = 1, \ldots, n_\mu$.
- The L^μ_α are generated by right multiplication with primitive idempotents e^μ_α.
- Having found all primitive idempotents, one can straightforwardly construct all irreps of the group.
- Exercises: Reduction of the regular rep of C_3.
- In Section 5 we will use this method in order to construct all irreps of S_n.

4.3.1 Dimensions and characters of the irreducible representations

Theorem 14. *Let G be a group with group algebra $\mathcal{A}(G)$, and let*

$$e_\mu = \sum_{g\in G} a_g\, g \qquad \left(a_g \in \mathbb{C}\,,\ \substack{\textit{linear combination}\\ \textit{of group elements}}\right)$$

be a primitive idempotent with corresponding left ideal $L^\mu = \mathcal{A}(G)e_\mu$, carrying the irreducible representation Γ^μ, $\dim \Gamma^\mu = d_\mu$. Then $\forall\, h \in G$

$$\chi^\mu(h) = \operatorname{tr} \Gamma^\mu(h) = \frac{|G|}{n_c} \sum_{g\in c} \overline{a_g}$$

where c is the conjugacy class of h with n_c elements.

Remark: $d_\mu = \chi^\mu(e) = |G|\overline{a_e}$.

Proof:
Define the linear map

$$A_h : \mathcal{A}(G) \ni r \mapsto h^{-1} r e_\mu\,.$$

(i) The trace of A_h is the character of h^{-1}:
Choose a basis $\{r_1, \ldots, r_{|G|}\}$ of $\mathcal{A}(G)$ s.t. $\{r_1, \ldots, r_{d_\mu}\}$ is a basis of L^μ. Then

$$A_h r_j = h^{-1} r_j e_\mu$$

contains no terms proportional to r_k with $k > d_\mu$, i.e. now $j \le d_\mu$,

$$A_h r_j = h^{-1} r_j e_\mu = h^{-1} r_j = \sum_{k=1}^{d_\mu} r_k \Gamma^\mu(h^{-1})_{kj}$$

and thus

$$\operatorname{tr} A_h = \chi^\mu(h^{-1}) = \overline{\chi(h)}$$

(w.l.o.g. choose Γ^μ unitary, all others equivalent).

(ii) Now choose the group elements $g \in G$ as basis for $\mathcal{A}(G)$. Then

$$\begin{aligned} A_h g = h^{-1} g e_\mu &= \sum_{g' \in G} a_{g'} \underbrace{h^{-1} g g'}_{\stackrel{?}{=} g \;\Leftrightarrow\; g' = g^{-1} h g} \\ &= a_{g^{-1}hg}\, g + \text{ terms not proportional to } g\,, \end{aligned}$$

and thus

$$\operatorname{tr} A_h = \sum_{g \in G} a_{g^{-1}hg} = \sum_{g' \in c} a_{g'} |G_{g'}| = \frac{|G|}{n_c} \sum_{g' \in c} a_{g'}\,,$$

where $G_{g'}$ is the stabiliser of g', and according the orbit-stabiliser theorem (see Problem 7) we have $n_c \cdot |G_{g'}| = |G|$.

Combining (i) and (ii) proves the theorem. □

5 Representations of the symmetric group and Young diagrams

The representation theory of S_n is fundamental in several ways:

- Finite groups of order n are isomorphic to subgroups of S_n (Theorem 1).
- Primitive idempotents in $\mathcal{A}(S_n)$ also play a role in the construction of irreps of classical Lie groups, as $\mathrm{U}(m)$, $\mathrm{O}(m)$ or $\mathrm{SU}(m)$.
- When considering quantum systems of identical particles S_n is always a "factor" of the symmetry group of the Hamiltonian H, i.e. the eigenstates of H transform in irreps of S_n.

5.1 One-dimensional irreducible representations and associate representations of S_n

The *alternating group* A_n is the group of even permutations of $\{1, 2, \ldots, n\}$ (i.e. each element is the product of an even number of transpositions). A_n is a normal subgroup of S_n, with quotient group $S_n/A_n \cong \mathbb{Z}_2$.

$\Rightarrow$ S_n has two one-dimensional representations, induced by the by the representations of $\mathbb{Z}_2$ (cf. Problems 10 & 16):

$$\Gamma^{\mathrm{s}}(p) = 1 \quad \forall\, p \in S_n \text{ (trivial representation) and}$$
$$\Gamma^{\mathrm{a}}(p) = \mathrm{sgn}(p) := \begin{cases} 1 & \text{for } p \text{ even} \\ -1 & \text{for } p \text{ odd} \end{cases} .$$

$\mathrm{sgn}(p)$ is called sign or parity of the permutation p.

Later: There are no other one-dimensional representations of S_n (see Section 5.5).

Alternatively, we obtain Γ^{s} and Γ^{a} from…

Lemma 15. *The symmetriser $s = \sum_{p \in S_n} p$ and the anti-symmetriser $a = \sum_{p \in S_n} \mathrm{sgn}(p) p$ are essentially idempotent and primitive.*

Proof: For s see remark after Theorem 13.

$$a^2 = \sum_{p,q} \mathrm{sgn}(p) p\, \mathrm{sgn}(q) q = \sum_p \underbrace{\sum_q \mathrm{sgn}(pq) pq}_{=a \text{ (rearrangement lemma)}} = n!\, a\,,$$

i.e. a is also essentially idempotent.

Representations: For all $q \in S_n$ we have

$$q\,ps = s = ps \quad \text{and}$$

$$q\,pa = \sum_r \text{sgn}(r)qpr = \text{sgn}(qp) \underbrace{\sum_r \text{sgn}(qpr)qpr}_{=a} = \text{sgn}(q)\,\text{sgn}(p)a = \text{sgn}(q)pa\,.$$

$\Rightarrow$ Both representations are one-dimensional, with matrix elements 1 and $\text{sgn}(q)$, respectively.

Remark: Non-equivalence can also be shown as follows: For all $p \in S_n$ we have

$$spa \underset{\substack{\uparrow \\ \text{rearrangement lemma: } sp = s}}{=} sa = \sum_{q,r} \text{sgn}(r)qr = \sum_q \text{sgn}(q) \underbrace{\sum_r \text{sgn}(qr)qr}_{= a \text{ (rearrangement lemma)}} = a \sum_q \text{sgn}(q) = 0\,.$$

$\Rightarrow$ s and a generate non-equivalent irreducible representations of S_n with basis vectors $\{ps\}$ and $\{pa\}$ $(p \in S_n)$, respectively.

Definition: (associate representations)

For a representation Γ^λ of S_n with dimension d_λ, we call Γ^λ and $\widetilde{\Gamma^\lambda} := \Gamma^\lambda \otimes \Gamma^{\text{a}}$ associate representations.

Remarks:

1. $\dim(\widetilde{\Gamma^\lambda}) = d_\lambda$
2. $\widetilde{\Gamma^\lambda}$ is irreducible $\Leftrightarrow$ Γ^λ is irreducible, since

$$\widetilde{\Gamma^\lambda}(p) = \text{sgn}(p)\Gamma^\lambda(p) \qquad \Rightarrow \qquad \sum_p |\widetilde{\chi^\lambda}(p)|^2 = \sum_p |\chi^\lambda(p)|^2$$

   ($= n!$ if irreducible).
3. If $\chi^\lambda(p) = 0$ for all odd p, then $\widetilde{\Gamma^\lambda}$ is equivalent to Γ^λ (since then all characters are identical, cf. Section 2.6), and Γ^λ is called *self-associate*. Otherwise they are non-equivalent.
4. Γ^{s} and Γ^{a} are associate to each other.

The following theorem is relevant for systems of bosons or fermions.

Theorem 16. *Let Γ^λ and Γ^μ be irreducible representations of S_n. Then*

(i) *$\Gamma^\lambda \otimes \Gamma^\mu$ contains Γ^{s} exactly once (not at all), if Γ^λ and Γ^μ are equivalent (non-equivalent).*

(ii) *$\Gamma^\lambda \otimes \Gamma^\mu$ contains Γ^{a} exactly once (not at all), if Γ^λ and Γ^μ are associate (not associate).*

Proof:

First: Consider only unitary representations of S_n
(all others are equivalent to unitary reps, cf. Theorem 2)
$\Rightarrow$ Characters of irreducible representations are real, since
p^{-1} is in the same conjugacy class as $p \Rightarrow \chi(p) = \chi(p^{-1}) \underset{\substack{\uparrow \\ \text{rep is unitary}}}{=} \overline{\chi(p)}$.

(i) Let a_s be the multiplicity of Γ^s in $\Gamma^{\lambda\otimes\mu}$.

$$a_\text{s} = \frac{1}{n!}\sum_p \underbrace{\overline{\chi^\text{s}(p)}}_{=1}\, \chi^{\lambda\otimes\mu}(p) = \frac{1}{n!}\sum_p \underbrace{\chi^\lambda(p)}_{=\overline{\chi^\lambda(p)}}\, \chi^\mu(p) = \begin{cases} 1 & \text{if } \Gamma^\lambda \text{ and } \Gamma^\mu \text{ are equivalent} \\ 0 & \text{otherwise} \end{cases} .$$

(ii) Let a_a be the multiplicity of Γ^a in $\Gamma^{\lambda\otimes\mu}$.

$$a_\text{a} = \frac{1}{n!}\sum_p \underbrace{\overline{\chi^\text{a}(p)}}_{=\text{sgn}(p)}\, \chi^{\lambda\otimes\mu}(p) = \frac{1}{n!}\sum_p \underbrace{\text{sgn}(p)\chi^\lambda(p)}_{=\widetilde{\chi^\lambda}(p)=\overline{\widetilde{\chi^\lambda}(p)}}\, \chi^\mu(p)$$

$$= \begin{cases} 1 & \text{if } \widetilde{\Gamma^\lambda} \text{ and } \Gamma^\mu \text{ equivalent, i.e. if } \Gamma^\lambda \text{ and } \Gamma^\mu \text{ associate} \\ 0 & \text{otherwise} \end{cases} .$$

□

5.2 Young diagrams and Young tableaux

Definition: (partition, Young diagram)
A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of a natural number n is a (finite) sequence of positive integers with

$$\sum_{i=1}^{r} \lambda_i = n \quad \text{and} \quad \lambda_i \geq \lambda_{i+1} .$$

Let λ and μ be two partitions for the same n.

(i) We say that λ and μ are equal, if $\lambda_i = \mu_i \;\forall\, i$.

(ii) We say $\lambda > \mu$ if the first non-vanishing term of the sequence $\lambda_i - \mu_i$ is positive.

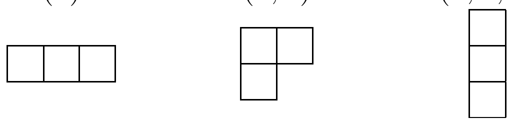
Graphically a partition can be represented as a *Young diagram*:

- n boxes, arranged in r rows, left-aligned,
- where the ith row consists of λ_i boxes.

Examples:

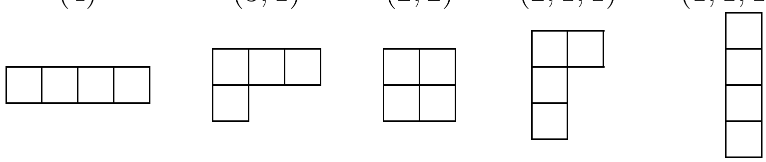
1. For $n = 3$ there are 3 different partitions:

(3) (2, 1) (1, 1, 1)



2. For $n = 4$ there are 5 different partitions:

(4) (3, 1) (2, 2) (2, 1, 1) (1, 1, 1, 1)



Remark: Each partition corresponds to a conjugacy class of S_n and vice versa:

- A conjugacy class is characterised by its cycle structure (see Problem 27).
- We read the ith row of the diagram as a λ_i-cycle.
- Each of the numbers $1, 2, \ldots, n$ appears in exactly one cycle $\Rightarrow$ $\sum_i \lambda_i = n$.

$\Rightarrow$ In particular, the number of Young diagrams for n is equal to the number of conjugacy classes of S_n, and thus equal to the number of non-equivalent irreducible representations of S_n.

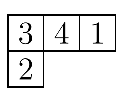
Example: For S_3 we have

$$\begin{aligned} \{e\} : &\quad \text{3 1-cycles, i.e } (1,1,1) \\ \{(12),(13),(23)\} : &\quad \text{1 2-cycle, 1 1-cycle, i.e. } (2,1) \\ \{(123),(132)\} : &\quad \text{1 3-cycle, i.e. } (3) \end{aligned}$$

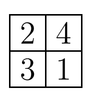
Further definitions:

- A *Young tableau* is a Young diagram, where each of the numbers $1, \ldots, n$ has been written into one of the boxes.

  Examples:

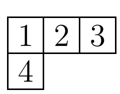
  

  or

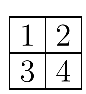
  

- In a *normal Young tableau* the numbers appear in increasing order, beginning in the first row from left to right, continuing in the second row etc.

  Examples:

  or

  For each Young diagram there is exactly one normal Young tableau.

- In a *standard Young tableau* the numbers increase in every row and every column (but not necessarily in strict order).

  Examples:

  $$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \text{or} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

- The normal Young tableau corresponding to the partition λ we denote by Θ_λ.
- We obtain an arbitrary tableau from Θ_λ by a permutation p of the n numbers in the boxes:

  $$\Theta_\lambda^p = p\Theta_\lambda\,.$$

  This implies $q\Theta_\lambda^p = \Theta_\lambda^{qp}$.

  Example:

  $$\Theta_{(2,2)}^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

Remark: The naming conventions in the literature vary, e.g. Young diagramm, Young graph, Young tableau, or Young frame.

5.3 Young operators

We will see that with each Young tableau we can associate a primitive idempotent generating a minimal left ideal in $\mathcal{A}(S_n)$ und thus an irrep of S_n.

Definitions: Let Θ_λ^p be a Young tableau.
A *horizontal* permutation h_λ^p permutes only numbers within rows of Θ_λ^p.
A *vertical* permutation v_λ^p permutes only numbers within columns of Θ_λ^p.
Furthermore, we define

the *(row-)symmetriser* $$s_\lambda^p = \sum_{\{h_\lambda^p\}} h_\lambda^p\,,$$

the *(column-)anti-symmetriser* $$a_\lambda^p = \sum_{\{v_\lambda^p\}} \mathrm{sgn}(v_\lambda^p)\, v_\lambda^p \quad \text{and}$$

the *Young operator* (or irreducible symmetriser) $$e_\lambda^p = s_\lambda^p\, a_\lambda^p = \sum_{\{h_\lambda^p\}} \sum_{\{v_\lambda^p\}} \mathrm{sgn}(v_\lambda^p)\, h_\lambda^p\, v_\lambda^p\,.$$

(Some books define $e = as$ instead of $e = sa$. This is only a matter of convention but leads to different intermediate results!)

Example: standard tableaux for S_3

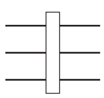
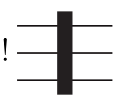
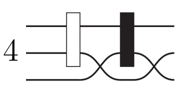
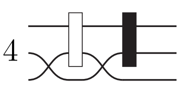
- $\Theta_1 := \Theta_{(3)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$: all p are horizontal: $s_1 = \sum_p p = s$ (symmetriser for S_3)
  only e is vertical: $a_1 = e$
  $e_1 = se = s$

- $\Theta_2 := \Theta_{(2,1)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$: e and (12) are horizontal: $s_2 = e + (12)$
  e und (13) are vertical: $a_2 = e - (13)$
  $e_2 = s_2 a_2 = e + (12) - (13) - (132)$

- $\Theta_3 := \Theta_{(1,1,1)} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$: only e is horizontal: $s_3 = e$
  all p are vertical: $a_3 = \sum_p \operatorname{sgn}(p) p = a$ (anti-symmetriser for S_3)
  $e_3 = ea = a$

- $\Theta_2^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$: e and (13) are horizontal: $s_2^{(23)} = e + (13)$
  e and (12) are vertical: $a_2^{(23)} = e - (12)$
  $e_2^{(23)} = s_2^{(23)} a_2^{(23)} = e - (12) + (13) - (123)$

In birdtracks: (cf. Section 1.4 and Problem 28)

$e_1 = 3!$  , $e_3 = 3!$  , $e_2 = 4$  , $e_2^{(23)} = 4$  .

Recall (see Problem 28) that open and solid bars over ℓ lines come with a normalisation factor of $1/\ell!$.

Observations:

Most of the general features (for S_n with n arbitrary) are already present in this example. (In the following we suppress the upper index p whenever that is unambiguous.)

1. For each tableau Θ_λ the horizontal and the vertical permutations, $\{h_\lambda\}$ and $\{v_\lambda\}$, form subgroups of S_n, with $\{h_\lambda\} \cap \{v_\lambda\} = \{e\}$.
   We obtain the subgroups for Θ_λ^p from those for Θ_λ by conjugation with p (which has the same effect as permuting the the numbers in the tableau); consequently $e_\lambda^p = p e_\lambda p^{-1}$. (In the birdtrack diagrams above we see this by intertwining the last two lines of e_2 on the left and on the right.)

2. s_λ and a_λ are (total) symmetriser and anti-symmetriser of the corresponding subgroup, in the sense that
$$s_\lambda h_\lambda = h_\lambda s_\lambda = s_\lambda \qquad \text{and} \qquad a_\lambda v_\lambda = v_\lambda a_\lambda = \operatorname{sgn}(v_\lambda) a_\lambda \,.$$

3. s_λ and a_λ are essentially idempotent, but in general not primitive.
   The e_λ are essentially idempotent and primitive (Exercises).

4. $e_1 = s$ and $e_3 = a$ generate the two one-dimensional irreps of S_3 (cf. Section 5.1).

e_2 generates a two-dimensional left ideal L_2 of $\mathcal{A}(S_3)$ (by right multiplication),

$$\begin{aligned} ee_2 &= e_2\,, \\ (12)e_2 &= (12) + e - (132) - (13) = e_2\,, \\ (23)e_2 &= (23) + (132) - (123) - (12) =: r_2\,, \\ (13)e_2 &= (13) + (123) - e - (23) = -e_2 - r_2\,, \\ (123)e_2 &= (123) + (13) - (23) - e = -e_2 - r_2\,, \\ (132)e_2 &= (132) + (23) - (12) - (123) = r_2\,, \end{aligned}$$

i.e. $L_2 = \mathrm{span}(e_2, r_2)$. Since e_2 is primitive, L_2 is minimal.
$\Rightarrow$ The Young operators of the normal Young tableaux generate all irreducible representations of S_3.

5. $e_2^{(23)}$ also generates an irreducible representation. It has to be equivalent to the irrep generated by e_2, since there are no more two-dimensional irreps of S_3.
The left ideal generated by $e_2^{(23)}$ is $L_2^{(23)} = \mathrm{span}(e_2^{(23)}, r_2^{(23)})$ with

$$r_2^{(23)} = (123) - (13) + (23) - (132)\,.$$

It is linearly independent from the other left ideals $L_1 = \mathrm{span}(e_1)$, $L_3 = \mathrm{span}(e_3)$, and L_2.

6. $\mathcal{A}(S_3)$ is the direct sum of these four minial left ideals.
The identity can be decomposed as

$$e = \frac{1}{6}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_2^{(23)} + \frac{1}{6}e_3\,,$$

i.e., the regular representation of S_3 is completely reduced by the Young operators corresponding to the standard Young tableaux.

5.4 Irreducible representations of S_n

Most observations about the Young operators for S_3 made in Section 5.3 carry over to S_n for arbitrary n. (The exception is Observation 6, which is only true for $n \leq 4$; it can be reestablished for $n \geq 5$ by modifying the Young operators.)

Theorem 17. *Let $\lambda \neq \mu$ be a partitions of $n \in \mathbb{N}$.*

(i) *The Young operators e_λ^p are essentially idempotent, i.e. $(e_\lambda^p)^2 = \eta_\lambda e_\lambda^p$ with $\eta_\lambda \neq 0$ and*

(ii) *the $\frac{1}{\eta_\lambda}e_\lambda^p$ are primitive idempotents.*

(iii) *The irreducible representations generated by e_λ and e_μ are not equivalent.*

(iv) *The irreducible representations generated by e_λ and e_λ^p are equivalent.*

Remark: The Young operators e_λ of the normal Young tableaux thus generate all non-equivalent irreps of S_n. ...since there are as many irreps as there are conjugacy classes and the conjugacy classes are labelled by partitions or Young diagrams.

Proof: First notice that no two terms in

$$e_\lambda = \sum_{\{h_\lambda\}} \sum_{\{v_\lambda\}} \operatorname{sgn}(v_\lambda) h_\lambda v_\lambda$$

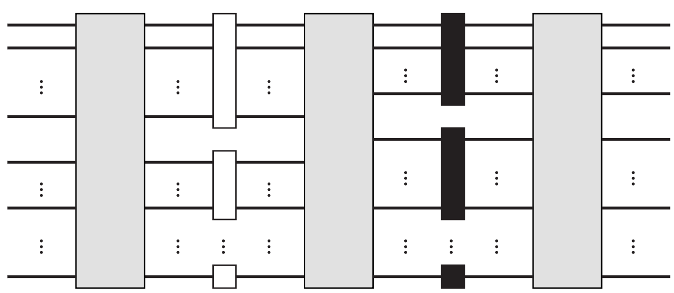
are the same, since

$$h_\lambda v_\lambda = h'_\lambda v'_\lambda \quad \Leftrightarrow \quad \underbrace{(h'_\lambda)^{-1} h_\lambda}_{\text{horizontal}} = \underbrace{v'_\lambda (v_\lambda)^{-1}}_{\text{vertical}} \quad \Leftrightarrow \quad h_\lambda = h'_\lambda \text{ and } v_\lambda = v'_\lambda$$

as $\{h_\lambda\} \cap \{v_\lambda\} = \{e\}$; in particular $e_\lambda \neq 0$ and

$$e_\lambda = e + \text{terms proportional to } p \in S_n \backslash \{e\} \, .$$

In birdtracks we have

$e^p_\lambda =$  . (∗)

- Within the grey boxes the lines are connected in some way (defined by the Young tableau Θ^p_λ).
- We also draw one-box (anti-)symmetrisers,

   ,

  i.e. each line in the middle is attached to exactly one symmetriser and one anti-symmetriser.
- The number of symmetrisers (anti-symmetrisers) is given by the number of rows (columns) of Θ_λ.
- The number of lines attached to a symmetriser (anti-symmetriser) is given by the number of boxes of the corresponding row (column).

Now all proofs will boil down to the question whether we can find a non-zero connection in the middle of diagrams like (∗).

(iii) We show $e_\lambda q e_\mu = 0 \; \forall \, q \in \mathcal{A}(S_n)$ (cf. Theorem 13): First observe that

$$e_\lambda q e_\mu = 0 \; \forall \, q \in \mathcal{A}(S_n) \qquad \Leftrightarrow \qquad e_\lambda p e_\mu = 0 \; \forall \, p \in S_n \, .$$

Since $e_\lambda p e_\mu = s_\lambda a_\lambda p s_\mu a_\mu$ we have a linear combination of terms of the form $s_\lambda p a_\lambda$, $p \in S_n$ which in birdtracks look like the diagram in $(*)$, but with the symmetrisers of e_λ on the left and the anti-symmetrisers of e_μ on the right. W.l.o.g. let $\lambda > \mu$.

The first (longest) symmetriser goes over λ_1 lines. For $s_\lambda p a_\lambda$ to be non-zero we have to connect each of these lines to a different anti-symmetriser, of which there are μ_1 many. If $\lambda_1 > \mu_1$ then at least two lines have to be connected to the same anti-symmetriser and the term vanishes.

If $\lambda_1 = \mu_1$ we continue with the second symmetriser: λ_2 lines which have to be connected to anti-symmetrisers that go over at least two lines – there are μ_2 many of these. If $\lambda_2 > \mu_2$ we get zero.

If $\lambda_2 = \mu_2$ we continue with the next symmetriser, but eventually we reach the first j s.t. $\lambda_j > \mu_j$.

(i) $(e^p_\lambda)^2 = s^p_\lambda a^p_\lambda s^p_\lambda a^p_\lambda$ is a linear combination of terms of the form $s^p_\lambda q a^p_\lambda$, $q \in S_n$. We already know that $s^p_\lambda q a^p_\lambda \neq 0$ for $q = e$ (since that's just e^p_λ). Varying q we get, by inspecting $(*)$,

- the same result, if q interchanges only lines which are attached to the same symmetriser,
- at most a sign if q interchanges only lines which are attached to the same anti-symmetriser,
- zero if q changes the way in which the symmetrisers and anti-symmetrisers are connected.

Thus, $(e^p_\lambda)^2 = \eta_\lambda e^p_\lambda$, but we still have to show that $\eta_\lambda \neq 0$. However, if η_λ was zero then e^p_λ would be nilpotent. Then the trace of the map $\mathcal{A}(S_n) \ni q \mapsto q e^p_\lambda$ would be zero, but the trace of this map is $n!$ (coefficient of e times the order of the group, cf. Section 4.3.1).

(ii) $e^p_\lambda q e^p_\lambda = s^p_\lambda a^p_\lambda q s^p_\lambda a^p_\lambda$ is again a linear combination of terms of the form $s^p_\lambda q a^p_\lambda$, $q \in S_n$; we have shown in (i) that they are all proportional to e^p_λ.

(iv) Since $e^p_\lambda = p e_\lambda p^{-1}$ we conclude that $e^p_\lambda p e_\lambda = p e_\lambda p^{-1} p e_\lambda \underset{\text{(i)}}{=} p \eta_\lambda e_\lambda \neq 0$.

□

Remark: Unfortunately, for $n \geq 5$ the Young operators for the standard tableaux no longer satisfy $e^p_\lambda e^q_\lambda = 0 \; \forall \, p \neq q$ (they still satisfy $e^p_\lambda e^q_\mu = 0 \; \forall \, \lambda \neq \mu$, see (iii) above). However, the ideals generated by the Young operators of the standard tableaux are still linearly independent (Exercises) and

$$\mathcal{A}(S_n) = \bigoplus_{\left\{\substack{\text{standard}\\ \text{tableaux}} \Theta^p_\lambda\right\}} \mathcal{A}(S_n) e^p_\lambda \,.$$

(without proof). In particular this implies that $\dim\left(\mathcal{A}(S_n) e^p_\lambda\right)$ is given by the number of standard tableaux for the partition λ.

5.5 Calculating characters using Young diagrams

The characters of the irreps of S_n, and in particular their dimensions $d_\mu = \chi^\mu(e)$, can be evaluated with the methods of Section 4.3.1. There are more efficient methods which we give here without proofs.

These methods are bases on the *Frobenius character formula* (or Frobenius-Weyl-Charakter-Formel) which relates characters of irreps of S_n to characters of irreps of S_m with $m < n$.

- The dimension d_λ of irrep Γ^λ with Young diagram Θ_λ is given by the number of standard tableaux for the partition Θ_λ. Two other formulas:

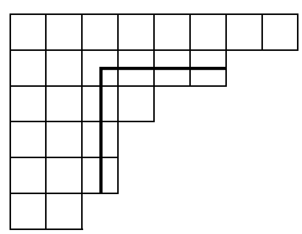
$$d_\lambda = n! \frac{\prod_{i<j}(\ell_i - \ell_j)}{\prod_i \ell_i!} = \frac{n!}{\prod_{i,k} h_{ik}}$$

  with

$$\begin{aligned} n! &= |S_n| \\ i, j &= 1, \ldots, r \; (r = \text{number of rows of } \Theta_\lambda) \\ k &= 1, \ldots, \lambda_i \; (\lambda_i = \text{number of boxes in row } i) \\ \ell_i &= \lambda_i + r - i \\ h_{ik} &= \text{number of boxes below and to the right of box } i, k + 1 \text{ for the box itself,} \\ &\quad\ \text{called the } \textit{hook length} \text{ of the box } i, k \end{aligned}$$

  Examples:

  (i)



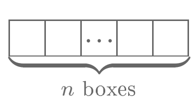
$$h_{23} = 7$$

  (ii) Young diagram with hook lengths written into the boxes:

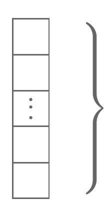
$$\Theta_\lambda = \begin{array}{|c|c|c|c|} \hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad \Rightarrow \quad d_\lambda = \frac{7!}{6 \cdot 4 \cdot 2 \cdot 1 \cdot 3 \cdot 1 \cdot 1} = 35$$

- This implies that S_n has only two one-dimensional irreps (Γ^{s} and Γ^{a}, cf. Section 5.1) with Young diagrams:

Γ^{s} :

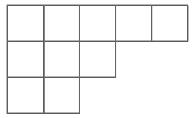
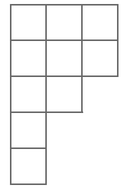


, Γ^{a} :

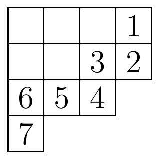


n boxes.

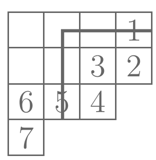
- For an irrep Γ^λ we obtain the associate irrep $\widetilde{\Gamma^\lambda}$ by transposing Θ_λ, i.e. by interchanging rows and columns:

$\Theta^\lambda =$  , $\widetilde{\Theta^\lambda} =$  .

- **Recursive evaluation of characters** of irreps of S_n:
  - The *boundary* of a Young diagramm is the right and lower boundary, i.e. a boundary field is any field, s.t. there is no field to the lower right of it.
    Example:



  - *skew-hook* := connected piece of the boundary, s.t. after removing this piece we retain a Young diagram.
    In the example above: 1–2, 1–4, 1–5, 1–7, 2, 2–4, 2–5, 2–7, 4, 4–5, 4–7, 7
    $\Rightarrow$ All end boxes of rows are starting boxes of skew hooks,
    all end boxes of columns are end boxes of skew hooks.
  - Each hook corresponds to a skew hook and vice versa.
    The hook length is equal to the length of the corresponding skew hook.
    Example: The skew hook 1–5 corresponds to the following hook:



  - A skew hook is called *positive* (*negative*), if the number of its vertical steps(= number of rows -1) is even (odd).
  - Let c be a conjugacy class of S_n with disjoint cycles of lengths $a_1, a_2, \ldots, a_q$.
    Wanted: character χ_c^λ of this class in irrep Γ^λ.
    * Choose any cycle of c, say with length a_i.
    * Denote by $\bar{c}$ the class of S_{n-a_i}, obtained by removing the cycle a_i from c.
    * For the Young diagram Θ_λ determine all skew hooks of length a_i and denote the Young diagram(s) of S_{n-a_i}, obtained by removing such a skew hook by $\Theta_{\bar{\lambda}}$. Then

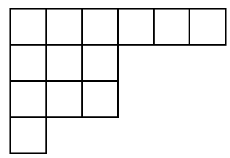
$$\chi_c^\lambda = \sum_{\bar{\lambda}} \pm \chi_{\bar{c}}^{\bar{\lambda}}$$

with "+" for positive skew hooks and "−" for negative skew hooks.

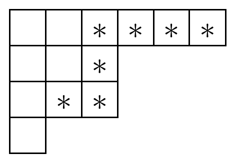
* Iterate this procedure.
* If no box of the Young diagram remains then $\chi^{\bar\lambda=0}_{()} = 1$.
  (Don't forget the sign of the last skew hook removed!)
* If there is no skew hook of length a_i then $\chi^\lambda_c = 0$.

This method is most efficient if we choose the cycle a_i s.t. there are as few skew hooks of length a_i as possible.

Examples:

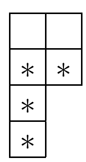
1. S_{13}, $c = (7,4,2)$, $\Gamma^\lambda = (6,3,3,1) =$ 

   – There is only one (skew) hook of length 7:

   $$\Rightarrow \quad \chi^{(6,3,3,1)}_{(7,4,2)} = +\chi^{(2,2,1,1)}_{(4,2)}$$

   – Now there is only one (skew) hook of length 4:

   $$\Rightarrow \quad \chi^{(6,3,3,1)}_{(7,4,2)} = +\chi^{(2)}_{(2)} = 1 \quad \text{(trivial rep)}$$

2. Once more, characters of the two-dimensional irrep of S_3, cf. Section 2.4.1 and Problem 29:

   $\chi^{(2,1)}_{(3)} = -1$ (remove completely, 1 vertical step)

   $\chi^{(2,1)}_{(2,1)} = 0$ (no skew hook of length 2)

   $\chi^{(2,1)}_{(1,1,1)} = \chi^{(1,1)}_{(1,1)} + \chi^{(2)}_{(1,1)} = 1 + 1 = 2$

6 Lie groups

When speaking about infinite groups we will combine the notion of a group with notions from others areas of mathematics. There will be precise definitions using notions like “topological space”, “connectedness” or “differentiable manifold”. However, we will not introduce all these notions and concepts in detail. If you are familiar with these notions – fine. If not, don’t panick! Some of the subtelties will not be relevant for the cases we are interested in, so we will gloss over them. Aspects which are important in our context will be introduced and discussed carefully, such that no prior knowledge beyond, say, multivariable calculus/analysis in $\mathbb{R}^n$ will be required.

6.1 Topological groups

Definition: (topological group)
A set G is called topological group if

(i) G (with some operation) is a group,

(ii) G is a topological space,

(iii) the map $G \ni g \mapsto g^{-1} \in G$ is continuous, and

(iv) the map $G \times G \ni (g, h) \mapsto gh \in G$ is continuous.

Examples:

1. Parametrise $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$ by the matrix elements $A_{ij} \in \mathbb{R}$, i.e. $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$, and choose on $\mathrm{GL}(n, \mathbb{R})$ the induced topology of (the standard topology of) $\mathbb{R}^{n^2}$.
   - The matrix elements of $C = AB$ are algebraic functions of A_{ij} and B_{kl}, i.e $(A, B) \mapsto AB$ is continuous.
   - $A \mapsto A^{-1}$ is also continuous, since the matrix elements of A^{-1} are rational, non-singular functions of the A_{jk}.

   $\Rightarrow$ $\mathrm{GL}(n, \mathbb{R})$ is a topological group.
2. By similar arguments $\mathrm{O}(n)$ or $\mathrm{SO}(n)$ topological groups as subsets of $\mathbb{R}^{n^2}$, and $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{U}(n)$ or $\mathrm{SU}(n)$ as subsets of $\mathbb{C}^{n^2}$.

Definition: (isomomorphism)
Two topological groups G and H are called isomorphic, if there exists a bijective map $f : G \to H$, which is both, an isomorphism of groups, and a homeomorphism of topological spaces (i.e. f is continuous and f^{-1} is continuous).

Example: The group $G_1 = (\mathbb{R}, +)$ is a topological group.
We define the group $G_2 = (\mathbb{R}, \oplus)$ by

$$x \oplus y = f\big(f(x) + f(y)\big)$$

where

$$f(x) = \begin{cases} x\,, & \text{if } x \le 1 \text{ or } x \ge 2 \\ 3 - x\,, & \text{if } 1 < x < 2 \end{cases}\,.$$

Notice that $f(f(x)) = x \;\forall x \in \mathbb{R}$. In G_2, for small $\varepsilon > 0$, we have $(1-\varepsilon)^{-1} = -1 + \varepsilon$, but $(1+\varepsilon)^{-1} = -2 + \varepsilon$, i.e. G_2 is not a topological group since property (iii) is violated. $f : G_2 \to G_1$ is an isomorphism of groups but **not** an isomorphism of topological groups.

Definition: (homogeneous space)
A topological space X is called homogeneous, if for every pair $x, y \in X$ there exists a homeomorphism $f : X \to X$ s.t. $f(x) = y$.

Remark: Every topological group G is homogeneous, since for any $g_1, g_2 \in G$ there is a (unique) $h \in G$ s.t. $g_2 = hg_1$ ($h = g_2 g_1^{-1}$). Thus, $f : g \mapsto hg$ is the desired homeomorphism (since group multiplication is continuous).

Homogeneity simplifies studying *local* properties dramatically: It is sufficient to study the group in a neighbourhood of one elements, e.g. in a neighbourhood of the identity.

Later, when we also can differentiate, then we can study local properties by expanding about the identity. This will lead us from Lie groups to Lie algebras.

Important *global* properties are *compactness* and *connectedness*. (disconnected, simply connected, multiply connected)

Examples (compactness):

1. Consider $\mathrm{O}(n) = \{A \in \mathbb{R}^{n\times n} \;:\; A^T A = \mathbb{1}\}$. The matrix elements A_{ij} of $A \in \mathrm{O}(n)$ satisfy

$$\sum_{k=1}^{n} A_{ik}A_{jk} = \delta_{ij} \qquad \Rightarrow \qquad \sum_{i,k=1}^{n} A_{ik}^2 = n\,,$$

   i.e. the elements of $\mathrm{O}(n)$ can be identified with points on sphere with radius $\sqrt{n}$ in $\mathbb{R}^{n^2}$. The union of these points is a closed[15] and bounded subset of this sphere and thus compact $\Rightarrow$ $\mathrm{O}(n)$ is compact.
   Similarly for $\mathrm{U}(n)$.

2. The Lorentz boosts Λ (transformations between coordinate systems with relative velocity v)

$$x_0' = \frac{x_0 - \frac{v}{c}x_1}{\sqrt{1 - \frac{v^2}{c^2}}}\,, \qquad x_1' = \frac{x_1 - \frac{v}{c}x_0}{\sqrt{1 - \frac{v^2}{c^2}}}\,, \qquad (c\text{: speed of light, } x_0 = c\cdot\text{time})$$

   form the group $\mathrm{O}(1,1)$ and as matrices can be parametrised as

$$\Lambda = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2} \qquad \text{with} \quad \beta = \tfrac{v}{c}\,.$$

[15] since it's the solution of a system of polynomial equations

Since $|v| < c$ we have $\beta \in (-1, 1)$, i.e. the parameter range is bounded but not closed $\Rightarrow$ the Lorentz group $\mathrm{O}(1,1)$ is not compact.
Maybe non-compactness es even more evident when using the parametrisation in terms of the rapidity t with $\beta = \tanh t$ (cf. Problem 11), since then $t \in \mathbb{R}$.

3. $\mathrm{GL}(n,\mathbb{R})$ is not compact because $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous but not bounded on $\mathrm{GL}(n,\mathbb{R})$ (since $|\det(\lambda A)| = |\lambda|^n\, |\det A|,\ \forall\ \lambda \in \mathbb{R}$).

Definition: (connected component)
The connected component of $g \in G$ is the union of all connected sets that contain g.

Remarks:

1. A connected component is actually connected.
2. (a) Let $G_0 \subseteq G$ be the connected component of the identity e.
   (b) If G is connected then $G_0 = G$.
   (c) If $G_0 = \{e\}$, then G is totally disconnected as due to homogeneity all other connected components then also contain just one element.
   (d) The connected component of g is $gG_0 = G_0 g$, since $g \in gG_0$ (and $\in G_0 g$) and since left and right multiplication are homeomorphisms and as such map connected sets to connected sets.
   (e) Hence G_0 is a normal subgroup.
   (f) The quotient group G/G_0 is totally disconnected, since $G/G_0 \cong \{gG_0 \,:\, g \in G\}$, i.e. for two different elements $h_1 G_0 \neq h_2 G_0$ (of the quotient group) h_2 cannot be contained in the connected component of h_1 (since this connected component is just the coset $h_1 G_0$).

Examples:

1. $\mathrm{SU}(2)$ is connected (even simply connected), since with the parametrisation of Problem 22,

$$\mathrm{SU}(2) \ni\ g = \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix}\,,$$
$$|u|^2 + |v|^2 = 1 \quad \Leftrightarrow \quad (\mathrm{Re}\,u)^2 + (\mathrm{Im}\,u)^2 + (\mathrm{Re}\,v)^2 + (\mathrm{Im}\,v)^2 = 1\,,$$

   $\mathrm{SU}(2)$ is homeomorphic to S^3, and spheres S^n with $n \geq 2$ are (simply) connected.

2. $\mathrm{O}(n)$ is not connected, since $O^T O = \mathbb{1}$ implies

$$1 = \det(OO^T) = (\det O)^2 \qquad \Leftrightarrow \qquad \det O = \pm 1$$

   i.e. $\mathrm{O}(n)$ has two connected components, $\mathrm{SO}(n) = \{O \in \mathrm{O}(n) \,:\, \det O = 1\}$ and $\{O \in \mathrm{O}(N) \,:\, \det O = -1\}$.

Before discussing Lie groups in general, let's look at an example which illustrates some of the basic ideas.

6.2 Example: SO(2)

- SO(2) = group of rotations in the plane $\mathbb{R}^2$ about the origin
- Parametrise by one parameter,
  natural choice: rotation angle ϕ with $0 \leq \phi < 2\pi$.
  (Any monotonous function of ϕ would also be finde.)
- Defining representation: action of SO(2) on vector in $\mathbb{R}^2$ (i.e. as an orthogonal 2×2 matrix)

$$x_j \mapsto \sum_k R_{jk} x_k \quad \text{with} \quad R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} . \tag{$*$}$$

- SO(2) is abelian, since $R(\phi_1)R(\phi_2) = R(\phi_1 + \phi_2) = R(\phi_2)R(\phi_1)$.
- Derivative:

$$\frac{\mathrm{d}R}{\mathrm{d}\phi}(\phi) = \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix}$$

  ...at the identity $\mathbb{1}$ ($\phi = 0$)

$$\frac{\mathrm{d}R}{\mathrm{d}\phi}(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =: -\mathrm{i}J \qquad \text{with} \qquad J = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} .$$

  (the factor $(-\mathrm{i})$ is physicists' convention)
  J is called *generator* of the group, since...

- Seek a differential equation of the form $\frac{\mathrm{d}R}{\mathrm{d}\phi} = AR$:

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}\phi}(\phi) &= \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} \underbrace{R(\phi)^{-1}}_{=R(-\phi)} R(\phi) \\ &= \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} R(\phi) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} R(\phi) = -\mathrm{i}JR(\phi) \end{aligned}$$

  Hence $R(\phi)$ solve the initial value problem $\frac{\mathrm{d}R}{\mathrm{d}\phi} = -\mathrm{i}JR$, $R(0) = \mathbb{1} \Rightarrow R(\phi) = \mathrm{e}^{-\mathrm{i}J\phi}$.

- With $J^2 = \mathbb{1}$ we have

$$\begin{aligned} R(\phi) = \mathrm{e}^{-\mathrm{i}J\phi} &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} J^n \phi^n \\ &= \sum_{n=0}^{\infty} \underbrace{\frac{(-\mathrm{i})^{2n}}{(2n)!} J^{2n}}_{=\frac{(-1)^n}{(2n)!}\mathbb{1}} \phi^{2n} + \sum_{n=0}^{\infty} \underbrace{\frac{(-\mathrm{i})^{2n+1}}{(2n+1)!} J^{2n+1}}_{=-\mathrm{i}\frac{(-1)^n}{(2n+1)!}J} \phi^{2n+1} \\ &= \mathbb{1}\cos(\phi) - \mathrm{i}J\sin\phi \, . \qquad \checkmark \text{ cf. } (*) \end{aligned}$$

- Viewed as a representation on $\mathbb{C}^2$ (although we introduced it as a representation on $\mathbb{R}^2$) the defining representation is reducible. It can be reduced by diagonalising J: $J = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ has eigenvalues ± 1 with eigenvectors $e_\pm = \begin{pmatrix} 1 \\ \pm \mathrm{i} \end{pmatrix}$, i.e.

$$Je_\pm = \pm e_\pm \qquad \Rightarrow \qquad R(\phi)e_\pm = \mathrm{e}^{\mp \mathrm{i}\phi} e_\pm \,,$$

  we find two one-dimensional (and thus irreducible) unitary representations, $\mathrm{e}^{\pm \mathrm{i}\phi}$.

- Consider now a (complex) vector space V, $\dim V = n$, and a representation of SO(2) in terms of unitary matrices $U(\phi)$ acting on V.
  We can write

$$U(\phi) = \mathrm{e}^{-\mathrm{i}J\phi}$$

  with a Hermitian $n \times n$ matrix J, since then

$$\begin{aligned} U(\phi_1)U(\phi_2) &= \mathrm{e}^{-\mathrm{i}J\phi_1}\mathrm{e}^{-\mathrm{i}J\phi_2} = \mathrm{e}^{-\mathrm{i}J(\phi_1+\phi_2)} \quad \text{(because the exponents commute)} \\ &= U(\phi_1+\phi_2) \qquad \text{and} \\ U(\phi)^\dagger &= \mathrm{e}^{\mathrm{i}J^\dagger\phi} = \mathrm{e}^{\mathrm{i}J\phi} = U(-\phi) = U(\phi)^{-1} \end{aligned}$$

  By diagonalising J we can completely reduce U $\Rightarrow$ all unitary irreducible representations are one-dimensional (also since SO(2) is abelian, cf. Problem 13).

- Now seek one-dimensional unitary representations, i.e. $J \in \mathbb{R}$. Due to $U(2\pi) = U(0)$ we demand

$$\mathrm{e}^{-2\pi \mathrm{i}J} = 1 \qquad \Leftrightarrow \qquad J = m \in \mathbb{Z}\,,$$

  i.e. the unitary irreducible representations $U^m(\phi) = \mathrm{e}^{-\mathrm{i}m\phi}$ are labelled by integers m:
  (i) $m = 0$: $R(\phi) \mapsto U^0(\phi) = 1$ (trivial representation)
  (ii) $m = 1$: $R(\phi) \mapsto U^1(\phi) = \mathrm{e}^{-\mathrm{i}\phi}$
  Isomorphism between SO(2) and the unit circle in $\mathbb{C}$, i.e. SO(2) $\cong$ U(1); thus everything observed for SO(2) is also true for U(1).
  (iii) $m = -1$: $R(\phi) \mapsto U^{-1}(\phi) = \mathrm{e}^{\mathrm{i}\phi}$,
  like (ii), but unit circle covered in opposite direction.
  (iv) $m = \pm 2$: $R(\phi) \mapsto U^{\pm 2}(\phi) = \mathrm{e}^{\mp 2\mathrm{i}\phi}$.
  Homomorphism SO(2) $\to U(1)$, with unit circle covered twice.
  Similarly for larger m.

  Only the representations with $m = \pm 1$ are faithful.

- Now consider $f : \mathrm{SO}(2) \to \mathbb{C}$ (sufficiently nice).
  Parametrising SO(2) by the rotation angle ϕ, f has to be a 2π-periodic function of ϕ. Then

$$\int_0^{2\pi} f(\phi)\, \frac{\mathrm{d}\phi}{2\pi}$$

is invariant under $\phi \mapsto \phi + \alpha$ for any fixed α; essentially, we integrate over SO(2), with normalisation chosen s.t. $|\mathrm{SO}(2)| = \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} = 1$.

With his we obtain: *Orthogonality* of representation matrices / characters (cf. Theorem 6 and corollary to Theorem 6),

$$\int_0^{2\pi} \overline{U^m(\phi)}\, U^n(\phi)\, \frac{\mathrm{d}\phi}{2\pi} = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-n)\phi}\, \frac{\mathrm{d}\phi}{2\pi} = \delta_{mn}\,,$$

and *completeness* (cf. Problem 19), i.e. the Fourier series of f,

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}n\phi}\, c_n = \sum_{n\in\mathbb{Z}} U^n(\phi)\, c_n \\ &\text{with} \qquad c_n = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}n\phi'} f(\phi')\, \mathrm{d}\phi' = \int_0^{2\pi} \overline{U^n(\phi')} f(\phi')\, \frac{\mathrm{d}\phi'}{2\pi}\,, \end{aligned}$$

converges to f (pointwise for, say, continuously differentiable f, otherwise at least in the L^2-sense),

Physics notation:

$$f(\phi) = \int_0^{2\pi} \underbrace{\frac{1}{2\pi} \sum_{n\in\mathbb{Z}} U^n(\phi)\, \overline{U^n(\phi')}}_{=\delta(\phi-\phi')} f(\phi')\, \mathrm{d}\phi'\,.$$

(δ-function/-comb as integral kernel of Fourier expansion)

6.3 Lie groups

Definition: (Lie group)

A set G is called Lie group, if:

(i) G is a group,

(ii) G is an analytic manifold,

(iii) the map $G \ni g \mapsto g^{-1} \in G$ is analytic, and

(iv) the map $G \times G \ni (g,h) \mapsto gh \in G$ is analytic.

Remarks:

1. An n-dimensional analytic manifold M is Hausdorff space equipped with charts (U_j, φ_j), i.e. $U_j \subseteq M$ open and homeomorphisms $\varphi_j : U_j \to \varphi(U_j) \subseteq \mathbb{R}^n$, with
   (i) $M = \bigcup_j U_j$ and
   (ii) $\varphi_j \circ \varphi_k^{-1} : \varphi_k(U_j \cap U_k) \to \varphi_j(U_j \cap U_k)$ analytic $\forall\ j, k$
   (i.e. can be expanded into convergent power series).
2. This means that locally the group elements are analytic functions of n parameters, where n is the dimension of G (as a manifold), more precisely:

   Consider a chart (U, φ) and $g, h, gh \in U$. Denote by x_j, $j = 1, \ldots, n$, the coordinates of g, and by y_j the coordinates of h, i.e.

$$\varphi(g) = (x_1, x_2, \ldots, x_n) = x \in \mathbb{R}^n$$
$$\varphi(h) = (y_1, y_2, \ldots, y_n) = y\,.$$

   Then the coordinates z_j of gh,

$$\varphi(gh) = (z_1, z_2, \ldots, z_n) = z\,,$$

   are analytic functions of x and y,

$$z_j = f_j(x, y)\,.$$

   Similarly, the coordinates of g^{-1} are analytic functions of x.
3. Now choose U with $e \in U$ and φ s.t. $\varphi(e) = 0 \in \mathbb{R}^n$, and f as above. Then

$$\begin{aligned} & f_j(x,0) = x_j\,, \qquad f_j(0,y) = y_j \\ &\text{and thus} \qquad \frac{\partial f_j}{\partial x_k}(0,0) = \frac{\partial f_j}{\partial y_k}(0,0) = \delta_{jk} \\ &\text{and also} \qquad \frac{\partial^2 f_j}{\partial x_k \partial x_l}(0,0) = \frac{\partial f_j}{\partial y_k \partial y_l}(0,0) = 0\,. \end{aligned}$$

   Expand $f(x, y)$ about $(0, 0)$,

$$f_j(x,y) = x_j + y_j + \sum_{k,l} \underbrace{\frac{\partial^2 f_j}{\partial x_k \partial y_l}(0,0)}_{=:a^j_{kl}}\, x_k y_l + \ldots$$

   and define

$$c^j_{kl} := a^j_{kl} - a^j_{lk}\,,$$

   the *structure constants of the Lie group* (coordinate dependent). They satisfy:
   (i) For abelian groups $c^j_{kl} = 0$, since then $f(x, y) = f(y, x)$.

(ii) $c^j_{kl} = -c^j_{lk}$

(iii) $\sum_l (c^j_{kl}c^l_{nm} + c^j_{nl}c^l_{mk} + c^j_{ml}c^l_{kn}) = 0$

The last identity follows from associativity of group multiplication by comparing the third order terms in the coordinate expansions of $g(h\tilde{g})$ and $(gh)\tilde{g}$.

Example: matrix Lie groups

1. Consider the matrix elements $A_{ij} \in \mathbb{R}$ of a group element $A \in \mathrm{GL}(n, \mathbb{R})$ as coordinates. The map
$$\psi : \mathbb{R}^{n^2} \to \mathbb{R}\,,\ A \mapsto \det A$$
is continuous, and thus the preimage $\psi^{-1}(0)$ of the closed set $\{0\}$ is closed. $\mathrm{GL}(n, \mathbb{R})$ is the complement of $\psi^{-1}(0)$ and hence open and an analytic submanifold of $\mathbb{R}^{n^2}$.
   - The matrix elements of $C = AB$ are algebraic functions of A_{ij} and B_{kl}, i.e $(A, B) \mapsto AB$ is analytic.
   - Likewise $A \mapsto A^{-1}$, since the matrix elements of A^{-1} are rational, non-singular functions of A_{jk}.

   Hence $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

2. For $\mathrm{GL}(n, \mathbb{C})$ consider real and imaginary part of the matrix elements as coordinates and argue as before (in terms of submanifolds of $\mathbb{R}^{2n^2}$).

3. For groups like $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$ or $\mathrm{SU}(n)$ one first observes that they are closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, respectively. One can show that closed subgroups of Lie groups are Lie (sub-)groups. (Later we will study some of these more explicitly.)

6.4 Lie algebras

Definition: A Lie algebra $\mathfrak{g}$ is a vector space over a field K (here mostly $\mathbb{R}$, sometimes $\mathbb{C}$), with an operation
$$\begin{aligned}[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g}\\ (X, Y) &\mapsto [X, Y]\end{aligned}$$
called *Lie bracket*, which satisfies ($\forall\ X, Y, Z \in \mathfrak{g}$):

(i) $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z] \qquad \forall\ \lambda, \mu \in K$ (linearity)

(ii) $[X, Y] = -[Y, X]$ (anti-symmetry)

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi identity)

Remarks:

1. A Lie algebra is called commutative if $[X, Y] = 0\ \forall\ X, Y \in \mathfrak{g}$.

2. One can show that the tangent space to a Lie group G at the identity is a Lie algebra $\mathfrak{g}$.

To this end consider curves $g(t)$ in G with $g(0) = e$. Then the derivative (in a chart) at $t = 0$ is a tangent vector.

For matrix Lie groups we can explicitly define the Lie algebra elements, also called generators, as matrices:

$$-\mathrm{i}\dot{g}(0) := -\mathrm{i}\frac{\mathrm{d}g}{\mathrm{d}t}(0) \in \mathfrak{g}\,.$$

The Lie bracket is now the matrix commutator (rather times $(-\mathrm{i})$, see below)

$$[X,Y] = XY - YX\,.$$

The commutator is linear and anti-symmetric, the Jacobi identity can be verified by direct calculation.

It remains to show that $X, Y \in \mathfrak{g}$ implies that also $(-\mathrm{i})[X,Y] \in \mathfrak{g}$.

To this end consider a curve $g(t)$ with $g(0) = e$, and thus $X := -\mathrm{i}\dot{g}(0) \in \mathfrak{g}$.

Define another curve $\tilde{g}(t) = h\,g(t)\,h^{-1}$ with $\tilde{g}(0) = heh^{-1} = e$, i.e.

$$-\mathrm{i}\dot{\tilde{g}}(0) = h\big(-\mathrm{i}\dot{g}(0)\big)h^{-1} = hXh^{-1} \in \mathfrak{g}\,.$$

With yet another curve $h(t)$ with $h(0) = e$, i.e. $Y := -\mathrm{i}\dot{h}(0) \in \mathfrak{g}$ define

$$\widetilde{X}(t) = h(t)\,X\,h(t)^{-1} \in \mathfrak{g}\,.$$

The derivative also takes values in $\mathfrak{g}$ (since $\mathfrak{g}$ is a vector space), and thus

$$\dot{\widetilde{X}}(0) = \mathrm{i}YX + X(\mathrm{i}Y) = -\mathrm{i}(XY - YX) = (-\mathrm{i})[X,Y] \in \mathfrak{g}\,.$$

Here we have used that $\frac{\mathrm{d}}{\mathrm{d}t}h(t)^{-1}\big|_{t=0} = -\mathrm{i}Y$, which follows from $\frac{\mathrm{d}}{\mathrm{d}t}\big(h(t)^{-1}h(t)\big) = 0$ and the product rule.)

Choosing a basis $\{X_j\}$ of $\mathfrak{g}$ we have

$$[X_j, X_k] = \mathrm{i}\sum_l c^l_{jk} X_l$$

with the *structure constants* c^l_{jk} *of the Lie algebra* (basis dependent).

The structure constants of the Lie algebra are equal to the structure constants of corresponding the Lie group (see Section 6.3) – supposing an appropriate choice of basis and coordinates: As basis $\{X_j\}$ for $\mathfrak{g}$ choose the tangent vectors to the coordinate lines in a chart $U \ni e$, i.e. for matrix Lie groups in an explicit parametrisation by taking derivatives with respect to the parameters,

$$X_j = -\mathrm{i}\dot{g}(0) \quad \text{with} \quad g(t) = \varphi^{-1}(0,\dots,0,x_j = t,0,\dots,0)\,,$$
$$\text{hence} \quad X_j = -\mathrm{i}\frac{\partial\varphi^{-1}}{\partial x_j}(0)\,.$$

In Section 6.3 we compared expansions of gh and hg, here we expanded $hgh^{-1} - g$. The properties (ii) & (iii) of the structure constants of Section 6.3 now follow from the Lie bracket properties (ii) & (iii) of the commutator.

3. It is sufficient to consider special curves, namely one-parameter subgroups, i.e. solutions of the initial value problem

$$\dot{g}(t) = \mathrm{i}X g(t)\,, \quad g(0) = e\,,$$

with $X \in \mathfrak{g}$. One writes $g(t) = \exp(\mathrm{i}Xt)$. For matrix Lie groups this exponential is given by the absolutely and uniformly convergent series

$$\exp(\mathrm{i}tX) = \sum_{\nu=0}^{\infty} \frac{(\mathrm{i}t)^\nu}{\nu!} X^\nu \qquad \text{(cf. Problem 33).}$$

For the special groups with $\det g = 1$ the generators are traceless, since

$$\det g(t) = \det(\mathrm{e}^{\mathrm{i}tX}) = \mathrm{e}^{\mathrm{i}t\,\mathrm{tr}\,X} \overset{!}{=} 1 \qquad \Leftrightarrow \qquad \mathrm{tr}\,X = 0\,.$$

For unitary groups, i.e. $gg^\dagger = \mathbb{1}$, the generators are Hermitian, since

$$g(t)^\dagger = g(t)^{-1} \quad \Leftrightarrow \quad \mathrm{e}^{-\mathrm{i}tX^\dagger} = \mathrm{e}^{-\mathrm{i}tX} \quad \Leftrightarrow \quad X = X^\dagger\,.$$

(See Problem 33 in both cases.)

Examples:

1. $G = \mathrm{SO}(3)$, i.e. rotations in 3 dimensions; defining representation in terms of 3×3 matrices R,

$$\vec{x} \mapsto R\vec{x}\,,$$

e.g. rotation by angle ϕ about the z-axis:

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Generator:

$$J_3 := J_z := -\mathrm{i}\frac{\mathrm{d}R_z}{\mathrm{d}\phi}(0) = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g} = \mathfrak{so}(3)$$

(Hermitian and traceless). Similarly for rotations about the x- or y-axis,

$$J_1 := J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix} \qquad \text{and} \qquad J_2 := J_y = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}.$$

One verifies by direct calculation that $[J_x, J_y] = \mathrm{i}J_z$ etc., i.e.

$$[J_j, J_k] = -\mathrm{i}\sum_{l=1}^{3} \varepsilon_{jkl}\, J_l$$

with the structure constants of SO(3) or $\mathfrak{so}(3)$:

$$\varepsilon_{jkl} = \begin{cases} 1\,, & j,k,l \text{ cyclic} \\ 0\,, & \text{at least 2 indices equal} \\ -1\,, & \text{otherwise} \end{cases} .$$

2. $G = \{O_A \text{ operators for rotations}\}$ (again, consider either as elements of some group G isomorphic to SO(3) or as a representation of SO(3)), acting on functions $f : \mathbb{R}^3 \to \mathbb{C}$ (cf. Section 2.4.1), say $f \in C^1(\mathbb{R}^3)$ as

$$(O_R f)(\vec{x}) = f(R^{-1}\vec{x}) \qquad \text{with } R \in \mathrm{SO}(3)\,.$$

Once more, rotation by angle ϕ about z-axis:

$$(O_{R_z(\phi)}f)(x,y,z) = f\left(R_z(\phi)^{-1}\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)\right) = f(x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi, z)\,.$$

Generator (viewed either as element of $\mathfrak{g}$ or as representation of an element of $\mathfrak{so}(3)$):

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\phi}(O_{R_z(\phi)}f)(x,y,z)\bigg|_{\phi=0} = -\mathrm{i}\left(\frac{\partial f}{\partial x}(\vec{x})\,y + \frac{\partial f}{\partial y}(\vec{x})\,(-x)\right) = \underbrace{\mathrm{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)}_{\in\mathfrak{g}} f(\vec{x})$$

In quantum mechanics $L_z = \frac{1}{\mathrm{i}}(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$ is the z-component of the so-called angular momentum operator $\vec{L} = \vec{x} \times (\frac{\hbar}{\mathrm{i}}\nabla)$ (here $\hbar = 1$). Commutators and structure constants as in the previous example.

Remark: In physics the generators typically are operators corresponding to quantities that can be measured (observables).

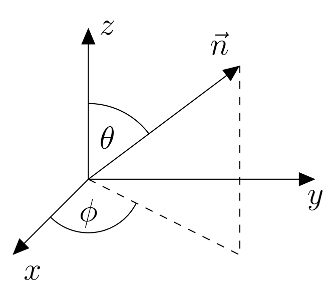
6.5 More on SO(3)

We study some global properties of SO(3) in terms of an explicit parametrisation.

- SO(3) = rotation group in 3 dimensions: 3 real parameters
  Consider, e.g., an orthogonal matrix $R \in \mathrm{SO}(3)$, consisting of 3 orthonormal columns: 1^{st} column, choose freely $\rightsquigarrow$ 2 parameters (angles – point on a 2-sphere), 2^{nd} second orthogonal to 1^{st} column, otherwise arbitrary $\rightsquigarrow$ 1 parameter (angle).

- We can parametrise rotations as $R_{\vec{n}}(\psi)$, with rotation angle ψ and rotation axis $\vec{n}$,

$$\vec{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} .$$

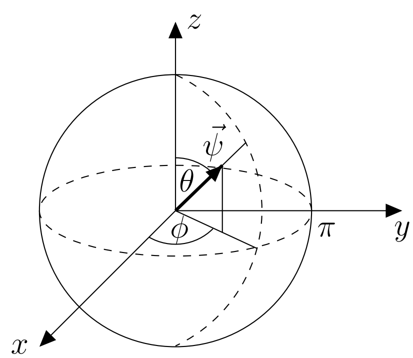


parameter ranges:

$$\begin{aligned} &0 \le \theta \le \pi \\ &0 \le \phi < 2\pi \\ &0 \le \psi \le \pi \qquad \text{(since we have } \vec{n} \text{ and } -\vec{n}\text{)} \end{aligned}$$

redundancies: (i) $R_{\vec{n}}(0) = R_{\vec{n}'}(0)$
(ii) $R_{\vec{n}}(\pi) = R_{-\vec{n}}(\pi)$

- A rotation thus corresponds to a vector $\vec{\psi} = \psi\vec{n}$, i.e. SO(3) corresponds to a ball in three-dimensional space with radius π.

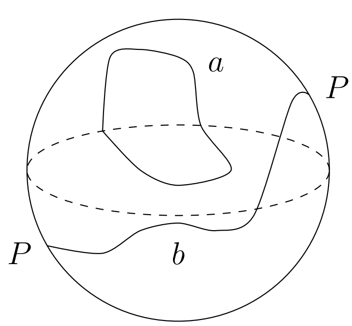


Using the cartesian components of $\vec{\psi}$ as parameters, $-\mathrm{i}\,\partial R/\partial\psi_j$ yields the generators of Section 6.4.

Back to the parametrisation in terms of $\theta, \phi, \psi \ldots$

This fixes redundancy (i), and due to redundancy (ii) antipodal points on the surface of the ball have to be identified (i.e. SO(3) is homeomorphic to the real projective space $\mathbb{R}P^3$).

- Consequently, there are two kinds of closed curves in SO(3): Curves which can be continuously contracted to a point, and curves for which this is not possible, i.e. SO(3) is connected but not simply connected.



Curve b is also closed in SO(3).

These global properties influence the possible representations of the group (as we will see later).

- Further observations:

  Rotations about a fixed axis form a (one-parameter) subgroup of SO(3). Such a subgroup is isomorphic to SO(2) (cf. Section 6.2).

  For arbitrary rotations $R \in \mathrm{SO}(3)$ we have (can be shown explicitly using the generators of Section 6.4)

$$RR_{\vec{n}}(\psi)R^{-1} = R_{\vec{n}'}(\psi) \qquad \text{with} \quad \vec{n}' = R\vec{n}\,.$$

  This implies that all rotations by the same angle are in the same conjugacy class.

Alternative parametrisation in terms of Euler angles
We just list some formulae; can be checked by direct computation.

- Every rotation can also be expressed in terms of Euler angles,

$$R = R_3(\alpha)R_2(\beta)R_3(\gamma)$$

  with

$$R_2(\psi) = R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} ,$$

$$R_3(\psi) = R_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

- parameter ranges:

$$0 \le \alpha, \gamma < 2\pi$$
$$0 \le \beta \le \pi$$

- relation with axis-angle parameters:

$$\phi = \frac{1}{2}(\pi + \alpha - \gamma)$$
$$\tan\theta = \frac{\tan\frac{\beta}{2}}{\sin\frac{\alpha+\gamma}{2}}$$
$$\cos\psi = 2\cos^2\frac{\beta}{2}\cos^2\frac{\alpha+\gamma}{2} - 1$$

6.6 Invariant integration: Haar measure

When representing finite groups we often used the rearrangement lemma as follows

$$\sum_{g\in G} f(g) = \sum_{g\in G} f(hg) = \sum_{g\in G} f(gh) \qquad \forall\, h \in G\,.$$

For continuous groups we would like to replace $\sum_{g\in G} f(g)$ by an integral, say, $\int_G f(g)\mathrm{d}\mu(g)$. To this end we need an invariant measure μ.

Theorem 18. (Haar measure)
Every compact topological group possesses a right- and left-invariant measure μ, called Haar measure; it is unique up to normalisation.

(in this generality without proof – but we will show explicitly how to construct μ for compact Lie groups)

Remarks:

1. Invariance means
$$\mu(gA) = \mu(Ag) = \mu(A)$$
$\forall\, g \in G$ and all Borel sets $A \subset G$, and in particular
$$\mathrm{d}\mu(gh) = \mathrm{d}\mu(hg) = \mathrm{d}\mu(g) \qquad \forall\, g, h \in G\,.$$

2. In the following for compact groups we normalise s.t.
$$\operatorname{vol} G = \int_G \mathrm{d}\mu(g) = 1\,.$$

3. Hence (e.g. for continuous functions)
$$\int_G f(hg)\,\mathrm{d}\mu(g) \underset{g'=hg}{=} \int_G f(g')\,\mathrm{d}\mu(h^{-1}g') = \int_G f(g')\,\mathrm{d}\mu(g') \qquad \text{and}$$
$$\int_G f(gh)\,\mathrm{d}\mu(g) \underset{g'=gh}{=} \int_G f(g')\,\mathrm{d}\mu(g'h^{-1}) = \int_G f(g')\,\mathrm{d}\mu(g')\,.$$

4. Moreover, $\int\limits_G f(g^{-1})\,\mathrm{d}\mu(g) = \int\limits_G f(g)\,\mathrm{d}\mu(g)$ or $\mathrm{d}\mu(g^{-1}) = \mathrm{d}\mu(g)$, since
$$\int_G f(g^{-1})\,\mathrm{d}\mu(g) = \int_G f(hg^{-1})\,\mathrm{d}\mu(g) = \int_G \underbrace{\int_G f(hg^{-1})\,\mathrm{d}\mu(h)}_{\int_G f(h)\,\mathrm{d}\mu(h)}\,\mathrm{d}\mu(g)$$
$$\underset{\int\limits_G \mathrm{d}\mu(g)=1}{=} \int_G f(h)\,\mathrm{d}\mu(h)\,.$$

5. **Uniqueness.** If μ and ν are both left- and right-invariant and normalised as $\int_G \mathrm{d}\mu(g) = \int_G \mathrm{d}\nu(g) = 1$, then $\mu = \nu$, since with
   (i) $\int_G f(g)\,\mathrm{d}\mu(g) = \int_G f(hg)\,\mathrm{d}\mu(g)$ and
   (ii) $\int_G f(f)\,\mathrm{d}\nu(h) = \int_G f(hg)\,\mathrm{d}\nu(h)$
   we can conclude that
$$\begin{aligned}\int_G\int_G f(hg)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(h) &\underset{\text{(i)}}{=} \int_G\int_G f(g)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(h) = \int_G f(g)\,\mathrm{d}\mu(g)\\ &\underset{\text{(ii)}}{=} \int_G\int_G f(h)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(h) = \int_G f(h)\,\mathrm{d}\nu(h)\,.\end{aligned}$$
6. One also finds invariant measures under weaker conditions, e.g. locally compact groups (like $\mathrm{GL}(n,\mathbb{R})$ or the Lorentz group) possess left-invariant and right-invariant measures (unique up to normalisation) but in general the two measures are not identical.

Many properties follow already from the existence of Haar measure – we don't have to know it explicitly. Nevertheless, let's continue with...

6.6.1 Calculating the Haar measure for a Lie group

Parametrise the group elements using $n = \dim G$ parameters, i.e.[16] $g = g(x_1,\dots,x_n)$, then (locally),
$$\mathrm{d}\mu(g) = \varrho(x_1,\dots,x_n)\,\mathrm{d}^n x$$
with a suitable density $\varrho(x)$ and Lebesgue measure $\mathrm{d}^n x = \mathrm{d}x_1\dots\mathrm{d}x_n$. We now construct ϱ s.t. invariance holds.

First: Behaviour of ϱ under reparametrisation (coordinate change/transition between different charts) $x = f(y)$:
$$\mathrm{d}\mu(g) = \varrho(x)\,\mathrm{d}^n x = \varrho(f(y))\underbrace{\left|\det\left(\frac{\partial f}{\partial y}(y)\right)\right|}_{\text{Jacobian}}\mathrm{d}^n y =: \tilde{\varrho}(y)\,\mathrm{d}^n y$$

Now expand $(-\mathrm{i})g(x)^{-1}\frac{\partial g}{\partial x_j}(x)$ in a basis $\{X_k\}$ of the Lie algebra $\mathfrak{g}$,
$$g(x)^{-1}\frac{\partial g}{\partial x_j}(x) = \mathrm{i}\sum_k X_k A(x)_{kj}$$

This is possible, because if $g(x) = e$ then the expression is a generator, else $\frac{\partial g}{\partial x_j}(x)$ lies in the tangent space at $g(x)$ and is transported to e by $g^{-1}(x)$.

[16] Actually $g = \varphi^{-1}(x_1,\dots,x_n)$ but we suppress chart-dependence for a moment.

Alternatively, explicitly consider $h(x,t) := g(x)^{-1} g(x + te_j)$, e_j a conical basis vector, for fixed x as curve in G. Then $h(x,0) = e$ and thus

$$\mathfrak{g} \ni \frac{\partial h}{\partial t}(x,0) = g(x)^{-1} \frac{\partial g}{x_j}(x)\,.$$

Claim: The density $\varrho(x) := |\det A(x)|$ defines a left-invariant measure.

Proof:

(i) First check behaviour under a change of coordinates $x = f(y)$. To this end denote $g(f(y)) =: \tilde{g}(y)$. We have

$$\begin{aligned}\tilde{g}(y)^{-1} \frac{\partial \tilde{g}}{\partial y_j}(y) &= g(f(y))^{-1} \sum_\ell \frac{\partial g}{\partial x_\ell}(f(y))\, \frac{\partial f_\ell}{\partial y_j}(y) \\ &= \mathrm{i} \sum_{\ell,k} X_k A(f(y))_{k\ell}\, \frac{\partial f_\ell}{\partial y_j}(y) \qquad \stackrel{!}{=} \mathrm{i} \sum_k X_k \tilde{A}(y)_{kj}\,,\end{aligned}$$

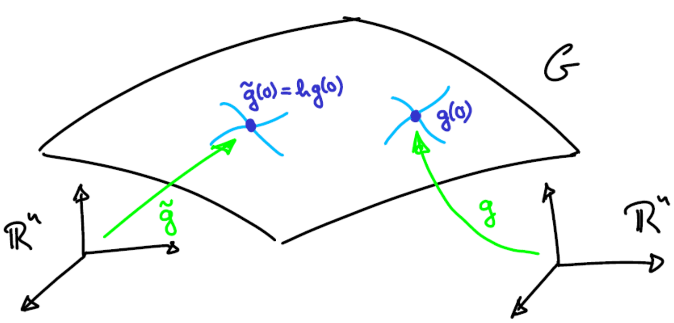
i.e. $\tilde{A}(y) = A(f(y))\, \frac{\partial f}{\partial y}(y)$ and thus

$$\tilde{\varrho}(y) = |\det \tilde{A}(y)| = \underbrace{|\det A(f(y))|}_{\varrho(f(y))} \left| \det \frac{\partial f}{\partial y}(y) \right|$$

as required.

(ii) Choose a special parametrisation (in a neighbourhood) of $\tilde{g} := hg$,

$$\tilde{g}(x) = h \cdot g(x)\,.$$



Then

$$\tilde{g}(x)^{-1} \frac{\partial \tilde{g}}{\partial x_j}(x) = (h \cdot g(x))^{-1} h \frac{\partial g}{\partial x_j}(x) = g(x)^{-1} \frac{\partial g}{\partial x_j}(x)$$

i.e. $\tilde{\varrho}(x) = \varrho(x)$ which implies the desired invariance,

$$\mathrm{d}\mu(hg) = \tilde{\varrho}(x)\, \mathrm{d}^n x = \varrho(x)\, \mathrm{d}^n x = \mathrm{d}\mu(g)\,.$$

(iii) Any other parametrisation can be achieved by further coordinate changes as in (i).

□

Now check right-invariance: Choose a parametrisation of $\tilde{g} := gh$ by

$$\tilde{g}(x) = g(x) \cdot h\,.$$

Then

$$\tilde{g}(x)^{-1}\frac{\partial \tilde{g}}{\partial x_j}(x) = h^{-1}g(x)^{-1}\frac{\partial g}{\partial x_j}(x)\,h = h^{-1}\,\mathrm{i}\sum_k X_k A(x)_{kj}\,h\,.$$

Since $h^{-1}X_k h \in \mathfrak{g}$,[17] we can write $h^{-1}X_k h = \sum_\ell X_\ell\,\varphi(h)_{\ell k}$ with a matrix $\varphi(h)$, i.e.

$$\tilde{g}(x)^{-1}\frac{\partial \tilde{g}}{\partial x_j}(x) = \mathrm{i}\sum_{k\ell} X_\ell\,\varphi(h)_{\ell k}A(x)_{kj} =: \mathrm{i}\sum_l X_\ell \tilde{A}(x)_{\ell j}$$

i.e. $\tilde{A}(x) = \varphi(h)A(x)$ and thus

$$\begin{aligned} \mathrm{d}\mu(gh) &= \tilde{\varrho}(x)\,\mathrm{d}^n x = |\det \tilde{A}(x)|\,\mathrm{d}^n x = |\det\varphi(h)|\,|\det A(x)|\,\mathrm{d}^n x \\ &= |\det\varphi(h)|\,\varrho(x)\,\mathrm{d}^n x = |\det\varphi(h)|\,\mathrm{d}\mu(g) \end{aligned}$$

The factor $|\det\varphi(h)|$ is called *modular function* of G. If $|\det\varphi(h)| = 1\ \forall\ h \in G$, we say that G is unimodular, and the left-invariant measure is also right-invariant.

Consider now

$$\int_G f(gh)\,\mathrm{d}\mu(g) \underset{g'=gh}{=} \int_G f(g')\,\mathrm{d}\mu(g'h^{-1}) = |\det\varphi(h^{-1})| \int_G f(g')\,\mathrm{d}\mu(g')$$

and for compact G choose the constant funktion $f \equiv 1$. Then

$$\int_G \mathrm{d}\mu(g) = |\det\varphi(h^{-1})| \int_G \mathrm{d}\mu(g)$$

i.e. compact Lie groups are unimodular.

Trivial example: SO(2) (cf. Section 6.2)
Parametrisation

$$g(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

generator

$$X = -\mathrm{i}\frac{\mathrm{d}g}{\mathrm{d}\phi}(0) = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix},$$

[17] Every Lie group acts by conjugation on its own Lie algebra (cf. Problems 38 & 40). Explicitly: Let $g(t)$ be a curve with $g(0) = e$ and $-\mathrm{i}\dot{g}(0) = X \Rightarrow \tilde{g}(t) = hg(t)h^{-1}$ is a curve with $\tilde{g}(0) = e$ and $-\mathrm{i}\dot{\tilde{g}}(0) = hXh^{-1}$, i.e. $hXh^{-1} \in \mathfrak{g}\ \forall\ h \in G$.

and thus

$$g(\phi)^{-1}\frac{\mathrm{d}g}{\mathrm{d}\phi}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathrm{i}X\,,$$

i.e. $A(\phi) = 1$ and hence $\mathrm{d}\mu(g) = \mathrm{d}\phi$ (as expected).

Now we proceed with what we can conclude already from the existence of the Haar measure (even before constructing it explicitly)

6.7 Properties of compact Lie groups

Theorems 2 and 6 (including the corollary) for representations of finite groups also hold for continuous representations of compact Lie groups, if in statements and proofs we replace

$$\frac{1}{|G|}\sum_{g\in G}\ldots \qquad \text{by} \qquad \int_G \ldots\,\mathrm{d}\mu(g)\,,$$

i.e.:

(i) Every finite-dimensional representation is equivalent to a unitary representation.

(ii) The matrix elements of unitary irreducible representations Γ^μ, Γ^ν (non-equivalent for $\mu \neq \nu$) are orthogonal, i.e.

$$\int_G \overline{\Gamma^\mu(g)_{jk}}\,\Gamma^\nu(g)_{j'k'}\,\mathrm{d}\mu(g) = \frac{1}{d_\mu}\delta_{\mu\nu}\delta_{jj'}\delta_{kk'}$$

with $d_\mu = \dim\Gamma^\mu$.

(iii) Similarly for the characters $\chi^\mu(g) = \operatorname{tr}\Gamma^\mu(g) = \sum_j \Gamma^\mu(g)_{jj}$,

$$\int_G \overline{\chi^\mu(g)}\,\chi^\nu(g)\,\mathrm{d}\mu(g) = \delta_{\mu\nu}\,.$$

This implies again:

$$\Gamma \text{ is irreducible} \quad \Leftrightarrow \quad \int_G |\chi(g)|^2\,\mathrm{d}\mu(g) = 1 \quad (\text{where } \chi(g) = \operatorname{tr}\Gamma(g))\,,$$

as well as: If Γ is a directe sum of irreducible representations, $\Gamma = \bigoplus_\mu a_\mu\Gamma^\mu$, then

$$a_\mu = \int_G \overline{\chi^\mu(g)}\,\chi(g)\,\mathrm{d}\mu(g)\,.$$

For finite groups we also showed completeness of the representation matrices' elements (cf. Problem 17) and the complete reducibility the regular representation, carried by the group algebra $\mathcal{A}(G)$ (cf. Section 4.3). This implied that there were only finitely many non-equivalent irreducible representation (see also Section 2.7).

Similarly one can show that compact Lie groups have countably many non-equivalent (continuous) irreducible representations, which are all of finite dimension. Moreover, every continuous representation is a direct sum of irreducible representations. All this follows from the *Peter-Weyl theorem.*

Consider the vector space $C(G)$ of continuous functions $\phi : G \to \mathbb{C}$ with scalar product

$$\langle\phi|\psi\rangle := \int_G \overline{\phi(g)}\, \psi(g)\, \mathrm{d}\mu(g)$$

(cf. the orthogonality relations for matrix elements and characters above). The role of the regular representation is assumed by Γ defined as

$$(\Gamma(h)\phi)(g) = \phi(h^{-1}g) \qquad \forall\, h \in G\,.$$

rep since

$$(\Gamma(h')(\Gamma(h)\phi))(g) = (\Gamma(h)\phi)(h'^{-1}g) = \phi(h^{-1}h'^{-1}g) = (\Gamma(h'h)\phi)(g)\,,$$

as for the O_A operators, cf., e.g., Section 2.4.1.

Theorem 19. (Peter-Weyl)
Let G be a compact Lie group with non-equivalent irreducible representations Γ^μ, $\dim\Gamma^\mu = d_\mu$. Then the matrix elements $\sqrt{d_\mu}\,\Gamma^\mu(g)_{jk}$, $j,k = 1,\ldots,d_\mu$, form a complete set of orthonormal functions for $C(G)$.

(without proof)

Remarks:

1. We can thus expand every function $f \in C(G)$ as

$$f(g) = \sum_{\mu,j,k} c_{\mu jk}\, \Gamma^\mu(g)_{jk}$$

   (convergence in L^2-sense) where

$$c_{\mu jk} = d_\mu \int_G \overline{\Gamma^\mu(g)_{jk}}\, f(g)\, \mathrm{d}\mu(g)\,.$$

   This generalises Fourier series (which we get for $\mathrm{SO}(2) \cong \mathrm{U}(1)$, cf. Section 6.2).

2. Completeness in physics notation:

$$\sum_{\mu,j,k} d_\mu\, \Gamma^\mu(g)_{jk}\, \overline{\Gamma^\mu(h)_{jk}} = \delta(g-h)$$

   with

$$\int_G \delta(g-h)\, f(g)\, \mathrm{d}\mu(g) = f(h)\,.$$

6.8 Irreducible representations of SO(3)

For every $g \in \mathrm{SO}(3)$ exists an $X \in \mathfrak{so}(3)$ s.t. $g = \mathrm{e}^{\mathrm{i}X}$. Choose, e.g., the basis

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

of $\mathfrak{so}(3)$ (generators from Section 6.4 times (-1)) with

$$[J_j, J_k] = \mathrm{i} \sum_\ell \varepsilon_{jk\ell} J_\ell \,.$$

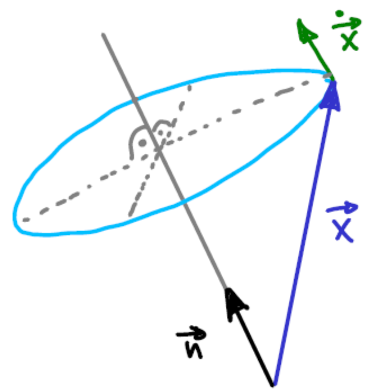
Then

$$R_{\vec{n}}(\psi) = \mathrm{e}^{-\mathrm{i}\psi \vec{n}\vec{J}} \qquad \text{where} \qquad \vec{n}\vec{J} = \sum_{j=1}^{3} n_j J_j$$

(rotation about axis $\vec{n}$ by angle ψ, cf. Section 6.5), since $\vec{x}(t) := \mathrm{e}^{-\mathrm{i}t\vec{n}\vec{J}}\vec{x}(0)$ solves

$$\dot{\vec{x}} = (-\mathrm{i}\vec{n}\vec{J})\,\vec{x} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -n_3 x_2 + n_2 x_3 \\ n_3 x_1 - n_1 x_3 \\ -n_2 x_1 + n_1 x_3 \end{pmatrix} = \vec{n} \times \vec{x}\,,$$

i.e. circular motion / rotation about axis $\vec{n}$.



- From every representation of a Lie group we obtain (by taking derivatives) a representation of the corresponding Lie algebra (in terms of matrices).
  With $g(t)$, $g(0) = e$, $\dot{g}(0) = \mathrm{i}X$ and a rep Γ of G define the derived rep $\mathrm{d}\Gamma$ of $\mathfrak{g}$ by

$$\mathrm{d}\Gamma(X) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma\big(g(t)\big)\Big|_{t=0}\,.$$

- From a representation of the Lie algebra $\mathfrak{so}(3)$ we obtain (by exponentiating) a representation of the group SO(3), if the global (topological) properties are satisfied.

The operator

$$J^2 := \sum_{j=1}^{3} J_j^2$$

commutes with all generators (and thus with every $X \in \mathfrak{so}(3)$):

$$\begin{aligned}
[J^2, J_k] &= \sum_j [J_j^2, J_k] = \sum_j (J_j[J_j, J_k] + [J_j, J_k]J_j) \\
&= \mathrm{i} \sum_{j,\ell} (J_j \varepsilon_{jk\ell} J_\ell + \underbrace{\varepsilon_{jk\ell} J_\ell J_j}_{=\varepsilon_{\ell kj} J_j J_\ell}) = \mathrm{i} \sum_{j,l} \underbrace{(\varepsilon_{jk\ell} + \varepsilon_{j\ell k})}_{=0} J_j J_\ell = 0\,.
\end{aligned}$$

J^2 is not in the Lie algebra; it is a so-called Casimir operator and an element of the enveloping algebra (see later). $[\cdot, \cdot]$ is the (matrix) commutator.

- This further implies $[J^2, g] = 0 \ \forall\ g \in \mathrm{SO}(3)$, since $g = \mathrm{e}^{\mathrm{i}X}$ with $X \in \mathfrak{so}(3)$.
- For representations all this also holds for the representation matrices of g, X, and J^2.
- If the representation is irreducible then according to Schur's Lemma (Theorem 4), the representation matrix of J^2 is a multiple of the identity matrix.

Now consider a representation (in general reducible) on a vector space V.

Shortened notation: Denote the representation matricex of g, X, J^2 also by g, X, J^2 (instead of $\Gamma(g)$, $\mathrm{d}\Gamma(X)$ etc.).

Construct irreducible subspaces (and thus irreducible representations) as follows:

- Choose a suitable starting vector.
- Generate an irreducible basis by repeatedly applying the generators.

Suitable starting vector: Joint (normalised) eigenvector of J^2 and J_3 (possible since $[J^2, J_3] = 0$), in Dirac notation

$$J_3|m\rangle = m|m\rangle$$

(Here we do not indicate the eigenvalue of J^2 when labelling the states, since for the moment we stay in fixed eigenspace of J^2. Later we will write $|jm\rangle$ instead of $|m\rangle$.

Define

$$J_\pm := J_1 \pm \mathrm{i}J_2\,.$$

Then

$$[J_\pm, J_3] = [J_1 \pm \mathrm{i}J_2, J_3] = -\mathrm{i}J_2 \pm \mathrm{i}(\mathrm{i}J_1) = \mp(J_1 \pm \mathrm{i}J_2) = \mp J_\pm$$

and thus

$$J_3(J_\pm|m\rangle) = (J_\pm J_3 - [J_\pm, J_3])|m\rangle = (J_\pm m \pm J_\pm)|m\rangle = (m+1)(J_\pm|m\rangle)\,,$$

i.e. either $J_\pm|m\rangle \propto |m\pm 1\rangle$ or $J_\pm|m\rangle = 0$.

Since the invariant subspace has to be finite dimensional this sequence has to terminate on both sides, say at $m = j$ and at $m = \ell$ with $j \geq \ell$,

$$\begin{aligned} J_3|j\rangle &= j|j\rangle\,, & J_3|\ell\rangle &= \ell|\ell\rangle\,, \\ J_+|j\rangle &= 0\,, & J_-|\ell\rangle &= 0\,. \end{aligned}$$

We further have

$$\begin{aligned} J_-J_+ &= (J_1 - \mathrm{i}J_2)(J_1 + \mathrm{i}J_2) = J_1^2 + J_2^2 + \mathrm{i}[J_1, J_2] \\ &= J_1^2 + J_2^2 - J_3 \qquad\qquad \Rightarrow \quad J^2 = J_3^2 + J_-J_+ + J_3 \end{aligned}$$

and

$$\begin{aligned} J_+J_- &= (J_1 + \mathrm{i}J_2)(J_1 - \mathrm{i}J_2) = J_1^2 + J_2^2 - \mathrm{i}[J_1, J_2] \\ &= J_1^2 + J_2^2 + J_3 \qquad\qquad \Rightarrow \quad J^2 = J_3^2 + J_+J_- - J_3\,. \end{aligned}$$

This implies

$$\begin{aligned} J^2|j\rangle &= (J_3^2 + J_3 + J_-J_+)|j\rangle = j(j+1)|j\rangle\,, \\ J^2|\ell\rangle &= (J_3^2 - J_3 + J_+J_-)|\ell\rangle = \ell(\ell-1)|\ell\rangle\,. \end{aligned}$$

Since all states lie in the same irreducible subspace, they are all in the same eigenspace of J^2, i.e.

$$j(j+1) = \ell(\ell-1)\,.$$

This is a quadratic equation with 2 solutions: $\ell = -j$ and $\ell = j+1$, but since $j \geq \ell$ we have

$$\ell = -j \qquad \text{and} \qquad j \geq 0\,.$$

Starting from ℓ we reach j with unit steps and thus

$$j - \ell = j - (-j) = 2j \in \mathbb{N}$$

Hence, $\mathfrak{so}(3)$ has irreducible representations with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

- The dimension of irrep j is $2j+1$.

- For orthonormal basis vectors, now denoted by $|jm\rangle$, we have

$$J^2|jm\rangle = j(j+1)|jm\rangle$$
$$J_3|jm\rangle = m|jm\rangle$$
$$J_\pm|jm\rangle = [j(j+1) - m(m\pm 1)]^{1/2}|j, m\pm 1\rangle$$

One obtains the last equation by calculating the norm of $J_\pm|m\rangle$.

Denote by $\Gamma^j(g)$ the *potential* representations of SO(3) defined by

$$\Gamma^j(g)|jm\rangle = g|jm\rangle\,,$$

i.e. the matrix elements are

$$\Gamma^j(g)_{mm'} = \langle jm|g|jm'\rangle\,,$$

and in particular

$$\Gamma^j(\mathrm{e}^{-\mathrm{i}tJ_3})_{mm'} = \langle jm|\mathrm{e}^{-\mathrm{i}tJ_3}|jm'\rangle = \langle jm|\mathrm{e}^{-\mathrm{i}tm'}|jm'\rangle = \mathrm{e}^{-\mathrm{i}tm}\delta_{mm'}\,.$$

We have $\mathrm{e}^{-2\pi\mathrm{i}J_3} = e$, but $\Gamma^j(\mathrm{e}^{-2\pi\mathrm{i}J_3})_{mm'} = \mathrm{e}^{-2\pi\mathrm{i}m}\delta_{mm'}$, i.e. only for

$$m\in\mathbb{Z} \qquad \Leftrightarrow \qquad j\in\mathbb{N}_0$$

do we have $\Gamma^j(\mathrm{e}^{-2\pi\mathrm{i}J_3}) = \mathbb{1}$ and only then we really get representations of SO(3).

Irreducible representations of SU(2)

The Pauli matrices σ_1, σ_2, σ_3 (cf. Problem 34) form a basis of the Lie algebra $\mathfrak{su}(2)$ with

$$[\sigma_j, \sigma_k] = 2\mathrm{i}\sum_l \varepsilon_{jkl}\sigma_l\,,$$

i.e. the matrices $\sigma_k/2$ satisfy the same relations as the J_k, and thus $\mathfrak{su}(2)\cong\mathfrak{so}(3)$. Hence we also already know all irreducible representations of $\mathfrak{su}(2)$. Since $\mathrm{SU}(2) = \exp(\mathrm{i}\mathfrak{su}(2))$ (Problem 37) and since SU(2) is simply connected, we get irreducible representations of SU(2) for all $j\in\mathbb{N}_0/2$.

Remark on the last step: According to Problem 38 the homomorphism $\varphi : \mathrm{SU}(2)\to\mathrm{SO}(3)$ satisfies $\varphi(\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}\vec{n}\vec{\sigma}}) = R_{\vec{n}}(\alpha)$, but $\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}\vec{n}\vec{\sigma}}$ is not the identity for $\alpha = 2\pi$. However, $\Gamma^j(\mathrm{e}^{-4\pi\mathrm{i}\frac{\sigma_3}{2}}) = \mathbb{1}_{2j+1}$ is true for every half-integer j.

Characters

Since all rotations by the same angle are in the same conjugacy class, is it sufficient to consider rotations about $\vec{e}_3$:

$$\chi^j(\psi) = \sum_{m=-j}^{j}\Gamma^j(R_{\vec{e}_3}(\psi))_{mm} = \sum_{m=-j}^{j}\mathrm{e}^{-\mathrm{i}m\psi} \qquad \text{for SO(3) with } j\in\mathbb{N}_0,\ \psi\in[0,\pi)\,,$$

$$\chi^j(\alpha) = \sum_{m=-j}^{j}\Gamma^j(\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}\sigma_3})_{mm} = \sum_{m=-j}^{j}\mathrm{e}^{-\mathrm{i}m\alpha} \qquad \text{for SU(2) with } j\in\mathbb{N}_0/2,\ \alpha\in[0,2\pi)\,.$$

In particular, for the *defining* (or "fundamental") representations

$$\chi^{1/2}(\alpha) = 2\cos(\tfrac{\alpha}{2})\,, \qquad \chi^1(\psi) = 1 + 2\cos\psi\,.$$

6.9 Remarks on some classical Lie groups

Definition: (adjoint representation)
Let G be Lie group with corresponding Lie algebra $\mathfrak{g}$, and let $g \in G$. The map $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g$ with

$$\begin{aligned} \mathrm{Ad}_g : \ \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto gXg^{-1} =: \mathrm{Ad}_g(X) \end{aligned}$$

is called adjoint representation of G (on $\mathfrak{g}$).

Remarks:

1. One also defines $\mathrm{Ad}_g(h) := ghg^{-1}$ for $h \in G$.
2. Ad is a representation since
   (i) $\mathfrak{g}$ is a vector space,
   (ii) $\mathrm{Ad}_g(X) \in \mathfrak{g}$, since $h(t) := g\mathrm{e}^{\mathrm{i}Xt}g^{-1}$ is a curve in G with $h(0) = e$ and $\dot{h}(0) = \mathrm{i}\mathrm{Ad}_g(X)$, and in particular
   $$g\mathrm{e}^{\mathrm{i}Xt}g^{-1} = \mathrm{e}^{\mathrm{i}\mathrm{Ad}_g(X)t}\,,$$
   (iii) $(\mathrm{Ad}_g \circ \mathrm{Ad}_h)(X) = \mathrm{Ad}_g(\mathrm{Ad}_h(X)) = \mathrm{Ad}_g(hXh^{-1}) = ghXh^{-1}g^{-1} = \mathrm{Ad}_{gh}(X)$
3. For $X \in \mathfrak{g}$ one further defines $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by
   $$\mathrm{ad}_X(Y) = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\mathrm{e}^{\mathrm{i}Xt}}(Y)\Big|_{t=0} = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\mathrm{i}Xt}Y\mathrm{e}^{-\mathrm{i}Xt}\right)\Big|_{t=0} = [X,Y]\,.$$

Lemma 20. (Principal axis theorem for unitary matrices)
For every $g \in \mathrm{U}(n)$ there exists an $h \in \mathrm{U}(n)$ s.t. $h^\dagger gh$ is diagonal, in particular

$$g = h \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{\mathrm{i}\varphi_n} \end{pmatrix} h^\dagger$$

with real φ_j.

Proof: Reduce to the principal axis theorem for Hermitian matrices.
Let $M_\phi := \{g \in \mathrm{U}(n) \,:\, \mathrm{e}^{\mathrm{i}\phi} \text{ is not eigenvalue of } g\}$. Then

$$\begin{aligned} f_\phi : M_\phi &\to \mathbb{C}^{n\times n} \\ g &\mapsto \mathrm{i}(\mathrm{e}^{\mathrm{i}\phi} + g)(\mathrm{e}^{\mathrm{i}\phi} - g)^{-1} \end{aligned}$$

(generalised *Cayley transformation*) maps unitary g to Hermitian matrices $A := f(g)$, since

$$\begin{aligned} A^\dagger &= (-\mathrm{i})(\mathrm{e}^{-\mathrm{i}\phi} - g^\dagger)^{-1}(\mathrm{e}^{-\mathrm{i}\phi} + g^\dagger) \\ &= (-\mathrm{i})\underbrace{(\mathrm{e}^{\mathrm{i}\phi} + g)(\mathrm{e}^{\mathrm{i}\phi} + g)^{-1}}_{=\mathbb{1}}(\mathrm{e}^{-\mathrm{i}\phi} - g^\dagger)^{-1}(\mathrm{e}^{-\mathrm{i}\phi} + g^\dagger) \\ &= (-\mathrm{i})(\mathrm{e}^{\mathrm{i}\phi} + g)(\mathbb{1} - \mathrm{e}^{\mathrm{i}\phi}g^\dagger + \mathrm{e}^{-\mathrm{i}\phi}g - \mathbb{1})^{-1}(\mathrm{e}^{-\mathrm{i}\phi} + g^\dagger) \\ &= \mathrm{i}(\mathrm{e}^{\mathrm{i}\phi} + g)\underbrace{(\mathrm{e}^{\mathrm{i}\phi}g^\dagger - \mathrm{e}^{-\mathrm{i}\phi}g)^{-1}(\mathrm{e}^{-\mathrm{i}\phi} + g^\dagger)}_{=:B} \end{aligned}$$

and

$$\begin{aligned} B(\mathrm{e}^{\mathrm{i}\phi} - g) &= (\mathrm{e}^{\mathrm{i}\phi}g^\dagger - \mathrm{e}^{-\mathrm{i}\phi}g)^{-1}(\mathrm{e}^{-\mathrm{i}\phi} + g^\dagger)(\mathrm{e}^{\mathrm{i}\phi} - g) \\ &= (\mathrm{e}^{\mathrm{i}\phi}g^\dagger - \mathrm{e}^{-\mathrm{i}\phi}g)^{-1}(\mathbb{1} + \mathrm{e}^{\mathrm{i}\phi}g^\dagger - \mathrm{e}^{-\mathrm{i}\phi}g - \mathbb{1}) = \mathbb{1}\,, \end{aligned}$$

i.e. $A^\dagger = A$. Now there exists an $h \in \mathrm{U}(n)$ s.t. $h^\dagger A h = D$ is diagonal (principal axis theorem for Hermitian matrices). Furthermore, f_ϕ is bijective (as function from M_ϕ to the Hermitian $n \times n$ matrices) with

$$\begin{aligned} && A &= \mathrm{i}(\mathrm{e}^{\mathrm{i}\phi} + g)(\mathrm{e}^{\mathrm{i}\phi} - g)^{-1} \\ \Leftrightarrow && A(\mathrm{e}^{\mathrm{i}\phi} - g) &= \mathrm{i}(\mathrm{e}^{\mathrm{i}\phi} + g) \\ \Leftrightarrow && \mathrm{e}^{\mathrm{i}\phi}(A - \mathrm{i}) &= (A + \mathrm{i})g \\ \Leftrightarrow && g &= \mathrm{e}^{\mathrm{i}\phi}(A + \mathrm{i})^{-1}(A - \mathrm{i}) = f^{-1}(A)\,. \end{aligned}$$

Now, for a given $g \in \mathrm{U}(n)$ choose ϕ s.t. $g \in M_\phi$, call $A := f_\phi(g)$, and choose $h \in \mathrm{U}(n)$ s.t. $h^\dagger A h =: D$ is diagonal. Then h also diagonalises g:

$$h^\dagger g h = h^\dagger \mathrm{e}^{\mathrm{i}\phi}(A + \mathrm{i})^{-1} h h^\dagger (A - \mathrm{i}) h = \mathrm{e}^{\mathrm{i}\phi}(D + \mathrm{i})^{-1}(D - \mathrm{i})\,.$$

□

Remark: The analogous result also holds for $g \in \mathrm{SU}(n) \subset \mathrm{U}(n)$, with $h \in \mathrm{SU}(n)$, since if $\det h \neq 1$, choose $\tilde{h} = (\det h)^{-\frac{1}{n}}\, h$ instead.

Theorem 21. *For every $g \in \mathrm{U}(n)$ there exists an $X \in \mathfrak{u}(n)$ s.t. $g = \mathrm{e}^{\mathrm{i}X}$.*

Proof: According to Lemma 20 there exists an $h \in \mathrm{U}(n)$ s.t.

$$g = h \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{\mathrm{i}\varphi_n} \end{pmatrix} h^\dagger = h \mathrm{e}^{\mathrm{i}Y} h^\dagger$$

with

$$Y = \begin{pmatrix} \varphi_1 & & 0 \\ & \ddots & \\ 0 & & \varphi_n \end{pmatrix} \in \mathfrak{u}(n)\,.$$

Moreover,

$$g = h e^{iY} h^\dagger = e^{i\mathrm{Ad}_h(Y)}$$

i.e. the desired $X \in \mathfrak{u}(n)$ is given by $X = \mathrm{Ad}_h(Y)$. □

Remarks:

1. With the remark after Lemma 20 we also have: For every $g \in \mathrm{SU}(n)$ there exists an $X \in \mathfrak{su}(n)$, s.t. $g = e^{iX}$.
2. Similarly for $g \in \mathrm{SO}(2n)$: One first shows that there exists an $h \in \mathrm{SO}(2n)$ s.t.

$$g = h \begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_n \end{pmatrix} h^T$$

   with $R_j \in \mathrm{SO}(2)$. For $\mathrm{SO}(2n+1)$ the diagonal matrix has an additional row with a 1. Then also every $g \in \mathrm{SO}(n)$ can be written as e^{iX} with $X \in \mathfrak{so}(n)$.
3. In all these cases we can in principle construct irreps using the same strategy as in Section 6.8 for SO(3) or SU(2): First construct irreducible representations of the Lie algebra and by exponentiation (potential) reps of the group.
4. The diagonal matrices which appear in procedure are maximal abelian subgroups (so-called *maximal tori*) of the corresponding group.

6.10 More on Lie algebras and related topics

With the reasoning of Section 6.9 we know when we can go from irreps of a Lie algebra to irreps of the corresponding Lie group. This was the last step in the procedure of Section 6.8. In the previous steps we used properties of J^2. In the following we discuss more generally what happened in that step and mention a couple of relevant notions.

Definition: (representations of Lie algebras)
Let $\mathfrak{g}$ be a Lie algebra and V a vector space. A representation ϕ assigns to each $X \in \mathfrak{g}$ a linear map $\phi(X) : V \to V$ s.t.

$$\phi(i \underbrace{[X,Y]}_{\text{Lie bracket}}) = \underbrace{[\phi(X), \phi(Y)]}_{\text{commutator}} \qquad \forall\ X, Y \in \mathfrak{g}\,.$$

The i-decoration comes from our convention that $G = \exp(i\mathfrak{g})$.

Examples:

1. $\mathrm{ad} : \mathfrak{g} \ni X \mapsto \mathrm{ad}_X$ with $\mathrm{ad}_X(Y) = [X, Y]$ defines a representation of $\mathfrak{g}$ on $\mathfrak{g}$

$$\begin{aligned}\mathrm{ad}_X(\mathrm{ad}_Y(Z)) - \mathrm{ad}_Y(\mathrm{ad}_X(Z)) &= [X,[Y,Z]] - [Y,[X,Z]] \\ &= [X,[Y,Z]] + [Y,[Z,X]] \\ &\underset{\text{Jacobi identity}}{=} -[Z,[X,Y]] \\ &= [[X,Y],Z] \\ &= \mathrm{ad}_{[X,Y]}(Z) \qquad \forall\ Z \in \mathfrak{g}\,.\end{aligned}$$

In a basis $\{X_j\}$ of $\mathfrak{g}$ the matrix elements of the representation matrices are given by the structure constants:

$$\begin{aligned}\mathrm{ad}_{X_j}(X_k) &=: \mathrm{i}\sum_l X_l\,(\mathrm{ad}_{X_j})_{lk} \\ &= [X_j, X_k] = \mathrm{i}\sum_l c^l_{jk}\,X_l\,.\end{aligned}$$

2. From a rep Γ of a Lie group G we obtain (by differentiation) a rep $\mathrm{d}\Gamma$ of the Lie algebra $\mathfrak{g}$,

$$\mathrm{d}\Gamma(X) = \frac{1}{\mathrm{i}}\,\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(\mathrm{e}^{\mathrm{i}Xt})\Big|_{t=0}\,.$$

In this Section the i-convention for the exponentiation is not optimal. . .

Definition: (enveloping algebra)
Let $\mathfrak{g}$ be a Lie algebra with basis $\{X_j\}$. The enveloping algebra $E(\mathfrak{g})$ consists of formal polynomials in the generators

$$\sum_j a_j(\mathrm{i}X_j) + \sum_{jk} b_{jk}(\mathrm{i}X_j)(\mathrm{i}X_k) + \sum_{jkl} c_{jkl}(\mathrm{i}X_j)(\mathrm{i}X_k)(\mathrm{i}X_l) + \dots\,, \qquad a_j, b_{jk}, c_{jkl} \in \mathbb{R}\,,$$

where $\mathrm{i}X_j\mathrm{i}X_k$ and $\mathrm{i}X_k\mathrm{i}X_j + \mathrm{i}X_l$ have to be identified if $[\mathrm{i}X_j, \mathrm{i}X_k] = \mathrm{i}X_l$.

Remarks:

1. A representation ϕ of a Lie algebra then also yields a representation of the enveloping algebra (call it also ϕ), whereby the formal products and sums become matrix products and matrix sums.

2. A basis of the enveloping algebra is, e.g., given by those monomials in the generators for which the indices are non-decreasing from left to right – all other monomials can obtained by exploiting the Lie bracket. Examples for SU(2):

$$\begin{aligned}\sigma_2\sigma_1 &= \sigma_1\sigma_2 - [\sigma_1, \sigma_2] = \sigma_1\sigma_2 - 2\mathrm{i}\sigma_3 \\ \sigma_1\sigma_3\sigma_2 &= \sigma_1(\sigma_2\sigma_3 - [\sigma_2, \sigma_3]) = \sigma_1\sigma_2\sigma_3 - 2\mathrm{i}\sigma_1\sigma_1\end{aligned}$$

Definition: (Casimir operator)
$C \in E(\mathfrak{g})$ is called Casimir operator if C commutes with all elements of the enveloping algebra, i.e. if

$$[C, A] = 0 \quad \forall\, A \in E(\mathfrak{g})\,.$$

Example: $J^2 := J_1^2 + J_2^2 + J_3^2$ for SO(3) (cf. Section 6.8).

Remarks:

1. In particular a Casimir operator commutes with all $X \in \mathfrak{g} \subseteq E(\mathfrak{g})$.
2. This implies $\mathrm{e}^{\mathrm{i}X} C \mathrm{e}^{-\mathrm{i}X} = C \;\forall\, X \in \mathfrak{g}$, i.e. in the cases of Section 6.8 and 6.9, where $G = \exp(\mathrm{i}\mathfrak{g})$, we immediately conclude $gCg^{-1} = C \;\forall\, g \in G$.
3. $gCg^{-1} = C \;\forall\, g \in G$ is even true more generally, since one can show:
   - $\exp(\mathrm{i}\mathfrak{g})$ always contains a neighbourhood of the identity in G.
   - By taking (finite) products $\mathrm{e}^{\mathrm{i}X}\mathrm{e}^{\mathrm{i}Y}\mathrm{e}^{\mathrm{i}Z}\ldots$ one reaches all $g \in G_0$, the connected component of the identity.
4. If G is connected, then for representations (of the Lie group, the Lie algebra and the enveloping algebra) we thus have $[\mathrm{d}\Gamma(C), \Gamma(g)] = 0 \;\forall\, g \in G$, and according to Schur's Lemma (Theorem 4) it follows that for irreps $\mathrm{d}\Gamma(C)$ is a scalar multiple of $\mathbb{1}$.

In the exercise class we will discuss the Killing form and a method for finding one Casimir operator for groups like SU(n) or SO(n).

7 Tensor method for constructing irreducible representations of GL(N) and subgroups

7.1 Setting

In the following let V be complex vector space with $\dim V = N$, i.e. $V \cong \mathbb{C}^N$.

Define $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ factors}}$.

Form tensor products from $|v_j\rangle \in V$, $j = 1, \ldots, n$:

$$\bigotimes_{j=1}^{n} |v_j\rangle = |v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle \in V^{\otimes n} \, .$$

General $|v\rangle \in V^{\otimes n}$ are linear combinations of tensor products, and are called *tensors of rank n*.

- Representation Γ of $\mathrm{GL}(N)$ on $V^{\otimes n}$: Defining representation γ on each factor, $g \in \mathrm{GL}(N)$,

  $$\Gamma(g) \bigotimes_{j=1}^{n} |v_j\rangle = \bigotimes_{j=1}^{n} \gamma(g)|v_j\rangle \, ,$$

  continue by linearly to all of $V^{\otimes n}$ (i.e. $\Gamma = \gamma^{\otimes n}$).

- Representation D of S_n on $V^{\otimes n}$: $p \in S_n$,

  $$D(p)\big(|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle\big) = |v_{p^{-1}(1)}\rangle \otimes |v_{p^{-1}(2)}\rangle \otimes \cdots \otimes |v_{p^{-1}(n)}\rangle \, ,$$

  also continued by linearity to all of $V^{\otimes n}$.
  D extends to representation of $\mathcal{A}(S_n)$.

Evidently,

$$\Gamma(g)D(p)|v\rangle = D(p)\Gamma(g)|v\rangle$$

$\forall\ p \in S_n$ (and also $\in \mathcal{A}(S_n)$), $\forall\ g \in \mathrm{GL}(N)$ and $\forall\ |v\rangle \in V^{\otimes n}$.

Notation: Form now on, we omit Γ and D, i.e. we write, e.g.,

$$gp|v\rangle = pg|v\rangle \, .$$

In a basis... Choose a basis of V: $|j\rangle$, $j = 1, \ldots, N$.

Form a product basis of $V^{\otimes n}$:

$$|j_1\rangle \otimes \cdots \otimes |j_n\rangle =: |j_1 \ldots j_n\rangle \, , \qquad j_k = 1, \ldots, N \ (k = 1, \ldots, n) \, .$$

General element $|x\rangle \in V^{\otimes n}$:

$$|x\rangle = \sum_{j_1,\ldots,j_n=1}^{N} x_{j_1\ldots j_n}|j_1\ldots j_n\rangle \underset{\substack{\uparrow \\ \text{summation convention}}}{=} x_{j_1\ldots j_n}|j_1\ldots j_n\rangle\,.$$

Then, e.g., (with $p \in S_n$)

$$\begin{aligned} p|x\rangle &= x_{j_1\ldots j_n}|j_{p^{-1}(1)}\ldots j_{p^{-1}(n)}\rangle \\ &= x_{j_{p(1)}\ldots j_{p(n)}}|j_1\ldots j_n\rangle\,. \end{aligned}$$

7.2 Decomposition of $V^{\otimes n}$ into irreducible invariant subspaces with respect to S_n and GL(N)

7.2.1 Symmetry classes

- **Notation:** Let (as in Section 5)
  - Θ_λ^p be a Young tableau
  - e_λ^p the corresponding Young operator
  - $L_\lambda = \{re_\lambda\,;\, r \in \mathcal{A}(S_n)\}$ the minimal left ideal generated by e_λ
    (cf. Section 5.4: $e_\lambda = e_\lambda^e$. The other e_λ^p also generate minimal left ideals, and the corresponding irreps for fixed λ are equivalent.)
- **Goal:** In the following we will see:
  - For fixed $|v\rangle \in V^{\otimes n}$ the subspace

    $$\{r|v\rangle : r \in L_\lambda\} = \mathcal{A}(S_n)e_\lambda|v\rangle$$

    is invariant and irreducible with respect to S_n.
  - For fixed e_λ^p the subspace

    $$\{e_\lambda^p|v\rangle : |v\rangle \in V^{\otimes n}\} = e_\lambda^p V^{\otimes n}$$

    is invariant and irreducible with respect to GL(N).
  - We can choose a basis $|\lambda,\alpha,a\rangle$ of $V^{\otimes n}$ s.t.
    λ lables the so-called symmetry class, given by a Young diagram,
    α labels the irreducible invariant subspaces w.r.t. S_n,
    a labels the irreducible invariant subspaces w.r.t. GL(N).
- For a fixed Young tableau the $\{e_\lambda^p|v\rangle : |v\rangle \in V^{\otimes n}\}$ are called *tensors of symmetry* Θ_λ^p.
- For a fixed Young diagram $\{r|v\rangle : r \in L_\lambda\,,\ |v\rangle \in V^{\otimes n}\} = \mathcal{A}(S_n)e_\lambda V^{\otimes n}$ are called *tensors of symmetry class* λ.

- First consider the subspace $T_\lambda(\alpha) = \{r|\alpha\rangle : r \in L_\lambda\}$ for fixed α:
  $T_\lambda(\alpha)$ is either empty or

  (i) $T_\lambda(\alpha)$ is invariant and irreducible under S_n and

  (ii) the S_n irrep carried by $T_\lambda(\alpha)$ is given by the irrep carried by L_λ.

  Proof:

  (i) Let $|v\rangle \in T_\lambda(\alpha)$, then $\exists\, r \in L_\lambda$ s.t.

$$
\begin{aligned}
& |v\rangle = r|\alpha\rangle \\
\Rightarrow \quad & p|v\rangle = \underbrace{pr}_{\in L_\lambda}|\alpha\rangle \in T_\lambda(\alpha) \quad \forall\, p \in S_n\,,
\end{aligned}
$$

  i.e. $T_\lambda(\alpha)$ is invariant under S_n. ("irreducible" follows from (ii))

  (ii) Let $\{r_i\}$ be a basis of $L_\lambda \quad \Rightarrow \quad \{r_i|\alpha\rangle\}$ is a basis of $T_\lambda(\alpha)$.

  a) action of S_n on L_λ: $p \in S_n$,

$$
pr_i = r_j \Gamma^\lambda(p)_{ji}\,.
$$

  b) action of S_n on $T_\lambda(\alpha)$: $p \in S_n$,

$$
pr_i|\alpha\rangle = r_j \Gamma^\lambda(p)_{ji}|\alpha\rangle = r_j|\alpha\rangle \Gamma^\lambda(p)_{ji}\,.
$$

  $\Rightarrow$ The representation matrices on $T_\lambda(\alpha)$ are the same as on L_λ, and in particular $T_\lambda(\alpha)$ is irreducible.

7.2.2 Totally symmetric and totally anti-symmetric tensors

- Let $\Theta_{\lambda=s} = \boxed{}\boxed{}\boxed{\cdots}\boxed{}$, i.e. $e_s = s$ is the total symmetriser of S_n, L_s is one-dimensional.

  $\Rightarrow$ For given $|\alpha\rangle$ the subspace $T_s(\alpha)$ is one-dimensional = span($e_s|\alpha\rangle$).

  These tensors are *totally symmmetric* (in all indices).

  Each $T_s(\alpha)$ carries the trivial representation of S_n.

  Example: $N = 2$, $n = 3 \quad \Rightarrow \quad e_s = \frac{1}{6}[e + (12) + (13) + (23) + (123) + (132)]$
  There are 4 different totally symmmetric tensors:

$$
\begin{aligned}
e_s|111\rangle &= |111\rangle && =: |s,1,1\rangle \\
e_s|112\rangle &= \tfrac{1}{3}(|112\rangle + |121\rangle + |211\rangle) && =: |s,2,1\rangle \\
e_s|122\rangle &= \tfrac{1}{3}(|122\rangle + |212\rangle + |221\rangle) && =: |s,3,1\rangle \\
e_s|222\rangle &= |222\rangle && =: |s,4,1\rangle
\end{aligned}
$$

  We denote the space spanned by the tensors of symmetry class s by T'_s.

- Totally anti-symmetric tensors ($\lambda = a$) exist only for $n \leq N$, i.e. only up to rank N,

$$\Theta_{\lambda=a} = \begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\,,$$

since for $n > N$ every basis vector contains at least two identical indices, say $j_k = j_l$ in $|j_1 \dots j_n\rangle \Rightarrow$ anti-symmetrisation yields zero.

The S_n irrep on $T_a(\alpha)$ is sgn.

- **Example:** Tensors of rank 2 ($n = 2$) in N dimensions

$$\begin{aligned} e_s|ii\rangle &= |ii\rangle && i = 1, \dots, N \\ e_s|ij\rangle &= \frac{1}{2}(|ij\rangle + |ji\rangle) && i \neq j \end{aligned}$$

$\Rightarrow N + \frac{N(N-1)}{2} = \frac{1}{2}(N^2 + N)$ totally symmetric tensors.

$$\begin{aligned} e_a|ii\rangle &= 0 && i = 1, \dots, N \\ e_a|ij\rangle &= \frac{1}{2}(|ij\rangle - |ji\rangle) && i \neq j \end{aligned}$$

$\Rightarrow \frac{1}{2}(N^2 - N)$ totally anti-symmetric tensors (*one* for $N = 2$).

7.2.3 Tensors with mixed symmetry

As an example consider again tensors of rank $n = 3$ in $N = 2$ dimensions, and in particular

$$\Theta_{\lambda=\kappa} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \text{with} \quad e_\kappa = [e + (12)][e - (13)]$$

From Section 5.3 we know: $L_\kappa = \mathrm{span}(e_\kappa, (23)e_\kappa)$

- First we choose $|\alpha\rangle = |112\rangle$,

$$\begin{aligned} e_\kappa|112\rangle &= [e + (12)][|112\rangle - |211\rangle] \\ &= 2|112\rangle - |211\rangle - |121\rangle \quad =: |\kappa, 1, 1\rangle\,, \\ (23)e_\kappa|112\rangle &= (23)[2|112\rangle - |211\rangle - |121\rangle] \\ &= 2|121\rangle - |211\rangle - |112\rangle \quad =: |\kappa, 1, 2\rangle\,. \end{aligned}$$

Then $T_\kappa(1) := \mathcal{A}(S_3)e_\kappa|112\rangle = \mathrm{span}(|\kappa, 1, 1\rangle, |\kappa, 1, 2\rangle)$ is invariant and irreducible under S_3 (cf. Section 5.3).

- Now we choose $|\alpha\rangle = |221\rangle$. Then

$$\begin{aligned} e_\kappa|221\rangle &= 2|221\rangle - |122\rangle - |212\rangle] =: |\kappa, 2, 1\rangle\,, \\ (23)e_\kappa|221\rangle &= 2|212\rangle - |122\rangle - |221\rangle] =: |\kappa, 2, 2\rangle\,, \end{aligned}$$

is a basis for another 2-dimensional, irreducible invariant subspace $T_\kappa(2)$.

- $|\kappa, 1, 1\rangle$ and $|\kappa, 2, 1\rangle$ are tensors of symmetry Θ_κ and span the 2-dimensional subspace $T'_\kappa(1) := e_\kappa V^{\otimes 3}$.
  (i) $T'_\kappa(1)$ is invariant under GL(2), since $gp = pg\ \forall\, g \in \mathrm{GL}(2)$ and $\forall\, p \in S_3$ implies

$$ge_\kappa|v\rangle = e_\kappa \underbrace{g|v\rangle}_{\in V^{\otimes 3}} \in T'_\kappa(1)\,.$$

This argument required neither $n = 3$ nor $N = 2$, i.e. it is true in general.

  (ii) $T'_\kappa(1)$ is irreducible under GL(2).

Proof: We explicitly construct the representation matrices for $g \in \mathrm{GL}(2)$.

$$
\begin{aligned}
g|\kappa,1,1\rangle &= g(2|112\rangle - |211\rangle - |121\rangle) \\
&\quad \text{recall that } g|112\rangle = |ijk\rangle g_{i1}g_{j1}g_{k2} \text{ (sum over } i,j,k) \\
&= 2|ijk\rangle g_{i1}g_{j1}g_{k2} - |ijk\rangle g_{i2}g_{j1}g_{k1} - |ijk\rangle g_{i1}g_{j2}g_{k1} \\
&\quad 3 \times 8 = 24 \text{ terms} \\
&= |112\rangle \underbrace{(2g_{11}g_{11}g_{22} - g_{12}g_{11}g_{21} - g_{11}g_{12}g_{21})}_{=2g_{11}\det g} \\
&\quad + |211\rangle \underbrace{(2g_{21}g_{11}g_{12} - g_{22}g_{11}g_{11} - g_{21}g_{12}g_{11})}_{=-g_{11}\det g} \\
&\quad + |121\rangle \underbrace{(2g_{11}g_{21}g_{12} - g_{12}g_{21}g_{11} - g_{11}g_{22}g_{11})}_{=-g_{11}\det g} \\
&\quad + |221\rangle \underbrace{(2g_{21}g_{21}g_{12} - g_{22}g_{21}g_{11} - g_{21}g_{22}g_{11})}_{=-2g_{21}\det g} \\
&\quad + |122\rangle \underbrace{(2g_{11}g_{21}g_{22} - g_{12}g_{21}g_{21} - g_{11}g_{22}g_{21})}_{=g_{21}\det g} \\
&\quad + |212\rangle \underbrace{(2g_{21}g_{11}g_{22} - g_{22}g_{11}g_{21} - g_{21}g_{12}g_{21})}_{=g_{21}\det g} \\
&\quad \text{The remaining terms have to vanish since } T'_\kappa(1) \text{ is invariant under GL}(N). \\
&= \det g\,\big(|\kappa,1,1\rangle g_{11} + |\kappa,2,1\rangle(-g_{21})\big)
\end{aligned}
$$

Similarly one finds

$$g|\kappa,2,1\rangle = \det g\,\big(|\kappa,1,1\rangle(-g_{12}) + |\kappa,2,1\rangle g_{22}\big)\,.$$

Hence the representation matrices,

$$\Gamma^\kappa(g) = \det g \begin{pmatrix} g_{11} & -g_{12} \\ -g_{21} & g_{22} \end{pmatrix}\,,$$

are also $\in \mathrm{GL}(2)$ and every GL(2)-matrix shows up as $\Gamma^\kappa(g)$. If the representation was reducible, all $\Gamma^\kappa(g)$ would have a joint eigenvector – obviously they don't, and thus the representation is irreducible. □

- Similarly one finds: $|\kappa,1,2\rangle$ and $|\kappa,2,2\rangle$ are tensors of symmetry $\Theta_\kappa^{(23)}$ and span the 2-dimensional subspace $T'_\kappa(2) := e_\kappa^{(23)} V^{\otimes 3}$, which is also invariant and irreducible under GL(2) and carries a representation that is equivalent to that carried by $T'_\kappa(1)$.

- The direct sum of subspaces $T'_\kappa(a)$ $(a = 1,2)$ contains all tensors of symmetry class κ with $\Theta_\kappa = \yng(2,1)$.

- Complete reduction of the 8-dimensional space $V^{\otimes 3}$:
  (recall that $\Theta_s = \yng(3)$ and $\Theta_\kappa = \yng(2,1)$)

  $$\begin{aligned} V^{\otimes 3} &= \underbrace{T_s(1)\oplus T_s(2)\oplus T_s(3)\oplus T_s(4)} \oplus \underbrace{T_\kappa(1)\oplus T_\kappa(2)} && \leftarrow \text{invariant under } S_3 \\ &= \qquad\qquad\quad T'_s \qquad\qquad\quad \oplus \overbrace{T'_\kappa(1)\oplus T'_\kappa(2)} && \leftarrow \text{invariant under GL}(2) \end{aligned}$$

  T'_s carries a 4-dimensional irrep of GL(2); under S_3 it is the dericet sum of 4 one-dimensional subspaces, each carrying the trivial rep.
  As a convenient basis for $V^{\otimes 3}$ we can choose:
  - the 4 totally symmetric tensors $|s,\alpha,1\rangle$ with $\alpha = 1,\ldots,4$ from Section 7.2.2,
  - the 4 tensors $|\kappa,\alpha,a\rangle$ with $\alpha = 1,2$ and $a = 1,2$.

7.2.4 Complete reduction of $V^{\otimes n}$

The observations and results of Section 7.2.3 generalise as follows ($V \cong \mathbb{C}^N$ as before).

- $V^{\otimes n}$ can be completely decomposed into irreducible S_n-invariant subspaces,

  $$V^{\otimes n} = \bigoplus_\lambda \bigoplus_\alpha T_\lambda(\alpha)\,.$$

  The λ-sum is only over Young diagrams with at most N rows ($N = \dim V$), (cf. the discussion of totally anti-symmetric tensors in Section 7.2.2).

- A basis of $T_\lambda(\alpha)$ is given by the tensors $|\lambda,\alpha,a\rangle$ with $a = 1,\ldots,\dim(T_\lambda(\alpha))$.
  The basis tensors can be chosen s.t. the representations matrices for S_n on $T_\lambda(\alpha)$ are identical for all α (which belong to the to the same symmetry class λ:

  $$p|\lambda,\alpha,a\rangle = |\lambda,\alpha,b\rangle \underbrace{\Gamma^\lambda(p)_{ba}}_{\text{independent of } \alpha}$$

- The decomposition of $V^{\otimes n}$ into irreducible S_n-invariant subspaces also leads to a decomposition into irreducible GL(N)-invariant subspaces:
  - The subspaces $T'_\lambda(a)$, spanned by $|\lambda,\alpha,a\rangle$ with fixed λ and a, are invariant (see Section 7.2.3) and irreducible (without proof) under GL(N).

– The GL(N)-irreps carried by $T'_\lambda(a)$ for fixed λ do not depend on a, i.e. same Young diagram, different (standard) Young tableaux $\rightsquigarrow$ equivalent irreps.

Proof: Let $|x\rangle \in T_\lambda(\alpha) \subseteq T'_\lambda$. Then $\exists\, r \in \mathcal{A}(S_n)$ with

$$|x\rangle = re_\lambda|\alpha\rangle\,.$$

For every $g \in \mathrm{GL}(N)$ we have (since $gp = pg\ \forall p \in S_n$)

$$g(re_\lambda)|\alpha\rangle = (re_\lambda)g|\alpha\rangle \in T_\lambda(g\alpha) \subseteq T'_\lambda\,,$$

i.e. g does not change the symmetry class (we already knew this since $T'_\lambda = \bigoplus_a T'_\lambda(a)$ is invariant under GL(N)), and thus

$$g|\lambda,\alpha,a\rangle = |\lambda,\beta,b\rangle\Gamma^\lambda(g)_{(\beta b)(\alpha a)}$$

(summing over the index pair (βb) – summation convention).
Now we show that $\Gamma^\lambda(g)_{(\beta b)(\alpha a)}$ is diagonal in the indices (a,b).
Let $g \in \mathrm{GL}(N)$, $p \in S_n$:

$$gp|\lambda,\alpha,a\rangle = g|\lambda,\alpha,c\rangle D^\lambda(p)_{ca} = |\lambda,\beta,b\rangle\Gamma^\lambda(g)_{(\beta b)(\alpha c)}\,D^\lambda(p)_{ca}$$

and

$$pg|\lambda,\alpha,a\rangle = p|\lambda,\beta,c\rangle\Gamma^\lambda(g)_{(\beta c)(\alpha a)} = |\lambda,\beta,b\rangle D^\lambda(p)_{bc}\,\Gamma^\lambda(g)_{(\beta c)(\alpha a)}\,.$$

Due to $gp = pg$ the r.h.s.s are equal. For fixed α and β, instead of the Latin indices we write a matrix product:

$$\Gamma^\lambda(g)_{\beta\alpha}\,D^\lambda(p) = D^\lambda(p)\,\Gamma^\lambda(g)_{\beta\alpha}\,.$$

Since this is true $\forall\, p \in S_n$ we conclude with Schur's Lemma (Theorem 4) implies that $\Gamma^\lambda(g)_{\beta\alpha}$ is a scalar multiple of the identity, and thus i.e. $\Gamma^\lambda(g)_{(\beta c)(\alpha a)}$ is diagonal in the Latin indices. □

7.2.5 Dimensions of the GL(N)-representations

Essentially we already know the dimensions of the GL(N)-irreps: To each Young diagram Θ_λ corresponds an S_n-irrep D^λ and a GL(N)-irrep Γ^λ. For the S_n-irreps we can determine dimensions and multiplicities (within $V^{\otimes n}$) using the methods of Sections 4.3.1 and 5. According to the construction in Sections 7.2.1–7.2.4 the multiplicity of D^λ is equal to the dimension of Γ^λ and vice versa. Determining the dimensions in this way can be tedious, and there are several other algorithms and formulae. . .

Graphical rule: Consider a Young diagram, e.g. $\yng(4,2,1)$ (i.e. S_7), and the corresponding normal Young tableau

$$\Theta_\lambda = \begin{ytableau} 1 & 2 & 3 & 4 \\ 5 & 6 \\ 7 \end{ytableau} \,.$$

Apply the Young operator e_λ to $|i_1 \dots i_7\rangle$. ($i_k \in 1, 2, \dots, N$, in general $N \neq n$; here $n = 7$)

Question: Which starting vectors lead to linearly independent results?

Write the is into the Young diagram:

$$\begin{ytableau} i_1 & i_2 & i_3 & i_4 \\ i_5 & i_6 \\ i_7 \end{ytableau} \qquad (*)$$

It was $e_\lambda = s_\lambda a_\lambda$ (see Section 5.3), and hence

(i) $e_\lambda |i_1 \dots i_n\rangle = 0$ if in a column at least two numbers are identical.

(ii) $e_\lambda v_\lambda = \mathrm{sgn}(v_\lambda) e_\lambda$, and thus $e_\lambda v_\lambda |i_1 \dots i_n\rangle$ and $e_\lambda |i_1 \dots i_n\rangle$ are linearly dependent.

Therefore, it is sufficient to consider starting vectors $|i_1 \dots i_n\rangle$ for which the numbers in each column of $(*)$ are *increasing.*

Now choose the is s.t. the entries in each row are *non-decreasing.* (Here equal values are allowed!)

One can show:

(i) The $e_\lambda |i_1 \dots i_n\rangle$ obtained in this way are linearly independent.

(ii) $e_\lambda h_\lambda |i_1 \dots i_n\rangle$ is a linear combination of tensors already constructed.

Due to $h_\lambda e_\lambda = e_\lambda$ the $e_\lambda |i_1 \dots i_n\rangle$ are symmetric in all is that stand in the same row in $(*)$. This restricts the number of basis tensors that can be constructed from a fixed set $\{i_1, \dots, i_n\}$ of indices.

With these rules we can determine the dimensions of the GL(N)-irreps, e.g. we have for $N = 2$ (cf. Section 7.2.3)

$$\dim \Gamma^{\yng(2,1)} = 2 \qquad \text{and} \qquad \dim \Gamma^{\yng(3)} = 4\,,$$

since the allowed choices are

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}$$

$$\text{as well as} \quad \begin{ytableau} 1 & 1 & 1 \end{ytableau}\,, \quad \begin{ytableau} 1 & 1 & 2 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} 2 & 2 & 2 \end{ytableau}\,.$$

For $\yng(1,1,1)$ and $N = 2$ there is no allowed choice for the distribution of the numbers 1 and 2. (This is consistent with the fact that there are no anti-symmetric tensors with $n > N$, cf. Section 7.2.2.)

We also find $\dim \Gamma^{\square} = 2$ for GL(2), since $\boxed{1}$ and $\boxed{2}$, and in general

$$\dim \Gamma^{\square} = N \quad \text{for} \quad \mathrm{GL}(N)\,,$$

where we write $\Gamma^{\square}$ for the defining representation, i.e.

$$V^{\otimes n} = \underbrace{\square \otimes \cdots \otimes \square}_{n \text{ factors}}\,.$$

Finally we can express the result of Section 7.2.3 as

$$\begin{array}{ccccccccccc} \square & \otimes & \square & \otimes & \square & = & \square\square\square & \oplus & \begin{smallmatrix}\square\square\\ \square\end{smallmatrix} & \oplus & \begin{smallmatrix}\square\square\\ \square\end{smallmatrix} \\ 2 & \cdot & 2 & \cdot & 2 & = & 4 & + & 2 & + & 2\,, \end{array}$$

for $N = 2$! In the exercises we will also study $N = 3$ and higher.

The above method is convenient for fixed N. In the exercises we will see a method using birdtracks, which yields the dimensions as functions of N.

Further formulae for the dimensions of the GL(N)-irreps (without proofs):

$$\dim(\Gamma^{\lambda}) = \left(\prod_{k=1}^{N-1} \frac{1}{k!}\right) \det\Big[(\underset{\substack{\uparrow \\ \text{number of boxes in row } i \text{ of } \Theta_\lambda}}{\lambda_i} + N - i)^{N-j}\Big]_{i,j=1,\ldots,m} = \left(\prod_{k=1}^{N-1} \frac{1}{k!}\right) \prod_{i<j}^{N} (\lambda_i - \lambda_j - i + j)$$

$$= \prod_{ij} \frac{N + j - i}{\underset{\substack{\uparrow \\ \text{hook length of box } i,j \text{ (see Section 5.5)}}}{h_{ij}}} \qquad \begin{array}{l}\text{(product over all boxes of } \Theta_\lambda \\ \ \ i = \text{row index, } j = \text{column index)}\end{array}$$

Back to the example $V^{\otimes 3}$, $N = 2$:

$$\begin{aligned} \dim(\Gamma^{\square\square\square}) &= \det \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} = 4 \\ &= \frac{2+1-1}{3} \cdot \frac{2+2-1}{2} \cdot \frac{2+3-1}{1} = \frac{2}{3} \cdot \frac{3}{2} \cdot 4 = 4 \\ \dim(\Gamma^{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}) &= \det \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} = 2 \\ &= \frac{2+1-1}{3} \cdot \frac{2+2-1}{1} \cdot \frac{2+1-2}{1} = \frac{2}{3} \cdot 3 \cdot 1 = 2 \end{aligned}$$

Remark: Using the tensor method one can construct all(?) *polynomial representations* of GL(N), i.e. reps for which the elements of the representation matrix for $g \in \mathrm{GL}(N)$ are polynomials in the the matrix elements of g. There are also other reps of GL(N), e.g.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)\,, \quad \Gamma(g) = \begin{pmatrix} 1 & \log|ad - bc| \\ 0 & 1 \end{pmatrix}\,.$$

7.3 Irreducible representations of U(N) and SU(N)

The irreducible representations of GL(N) (read GL($N,\mathbb{C}$), with dimension $2N^2$ as a real manifold) from Section 7.2.4 also restrict to representations of subgroups, which do not need to be irreducible. They are, however, irreducible for U(N) (dimension N^2) and SU(N) (dimension N^2-1) but in general not for O(N) and SO(N).

Idea behind this:

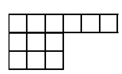
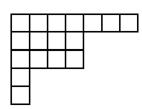
- The generators of GL(N) are the generators of U(N) complemented by i times the generators of U(N). $\rightsquigarrow$ If one can block-diagonalise the representation of the generators of U(N) one can also block-diagonalise the generators of the corresponding GL(N) rep.
- The generators of U(N) are the generators of SU(N) complemented by a multiple of the identity matrix. $\rightsquigarrow$ If one can block-diagonalise the representation of the generators of SU(N) one can also block-diagonalise the generators of the corresponding U(N) rep.

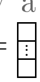
No such simple relation exists for O(N) or SO(N) (dimension $N(N-1)/2$ in both cases). Already for $V\otimes V$, which under GL(N) decomposes into symmetric and anti-symmetric tensors, the corresponding SO(N) rep on the symmetric subspace contains the trivial rep: Choose a basis $\{|j\rangle\}$ of V; then $|j\rangle\otimes|j\rangle$ (summation convention) is invariant under SO(N):

$$g\big(|j\rangle\otimes|j\rangle\big)=\big(|k\rangle\otimes|\ell\rangle\big)g_{kj}g_{\ell j}=\big(|k\rangle\otimes|\ell\rangle\big)\delta_{k\ell}=|k\rangle\otimes|k\rangle\ .$$

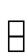
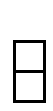
In the following we are interested in SU(N).

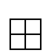
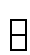
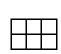
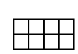
For SU(N) the two irreps corresponding to the Young diagrams (with row lenghts) $(\lambda_1,\dots,\lambda_N)$ and $(\lambda_1+k,\dots,\lambda_N+k)$ are equivalent, e.g.

 and  .

for $N=5$ and $k=1$. (Proof: see Problems 45 & 46.) (For GL(N) they differ by a factor of $(\det g)^k$, and $\det g=1$ for $g\in$ SU(N).) In particular, the Young diagram $\Theta_a=$  (N boxes) corresponds to the trivial representation, i.e. $g\mapsto 1\ \forall\, g\in$ SU(N). Tensors which transform under SU(N) in the trivial representation are called SU(N) scalars or SU(N) singlets. These tensors do, however, transform under S_n in the totally anti-symmetric rep (sgn).

Irreducible representations of SU(2)

- defining/fundamental representation: □, dimension 2
- trivial representation: , dimension 1
- $N=2\Rightarrow$ the Young diagrams have at most 2 rows, i.e. every irrep is equivalent to
  - either ,

e.g.  $\sim$  $\sim$  $\sim$ 

– or a one-row Young diagram, obtained by omitting all two-box columns,

e.g. 

⇒ Besides  we only have to one-row diagrams.

- Dimension of the irrep corresponding to a one-row diagram with k boxes:

$$\underbrace{\boxed{1\,|\cdots|\,1\,|\,1},\ \boxed{1\,|\cdots|\,1\,|\,2},\ \boxed{1\,|\cdots|\,2\,|\,2},\ \boxed{2\,|\cdots|\,2\,|\,2}}_{k+1 \text{ possibilities}}$$

...or using hook lengths:

$$\prod_{ij} \frac{N+j-i}{h_{ij}} = \prod_{j=1}^{k} \frac{2+j-1}{k-j+1} = \frac{(k+1)!}{k!} = k+1$$

⇒ For SU(2) there is exactly one irrep for each $k \in \mathbb{N}_0$, with dimension $k+1$ (cf. Section 6.8, where we arrived at the same result by different means.)

Irreps of SU(3)

- fundamental rep: , dimension 3
- triviale rep: , dimension 1
- $N = 3 \Rightarrow$ all Young diagrams have at most 3 rows, more precisely, all irreps are equivalent to either  or a diagram with at most 2 rows, i.e. $(\lambda_1, \lambda_2, 0)$ with

$$\dim(\Gamma^\lambda) = \frac{1}{2} \det \begin{pmatrix} (\lambda_1+2)^2 & \lambda_1+2 & 1 \\ (\lambda_2+1)^2 & \lambda_2+1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(\lambda_1+2)(\lambda_2+1)(\lambda_1-\lambda_2+1)\,.$$

7.4 Reducing tensor products in terms of Young diagrams

Given two irreps Γ^λ and $\Gamma^{\lambda'}$ of GL(N), U(N) or SU(N) with Young diagrams Θ_λ and $\Theta_{\lambda'}$.

Task: Completely reduce the product rep $\Gamma^\lambda \otimes \Gamma^{\lambda'}$.

`[examples motivating the following rules]`

From what we have learned so far one can deduce the following graphical rule (without proof):

1. Write the number i in all boxes of row i of $\Theta_{\lambda'}$.
2. Add the boxes of $\Theta_{\lambda'}$ to Θ_λ, in the first step the 1s, in the second step the 2s etc. adhering to the following rules:

(a) In each step the resulting diagram has to be a valid Young diagram and must not have more than N rows.

(b) A number may not appear more than once in the same column.

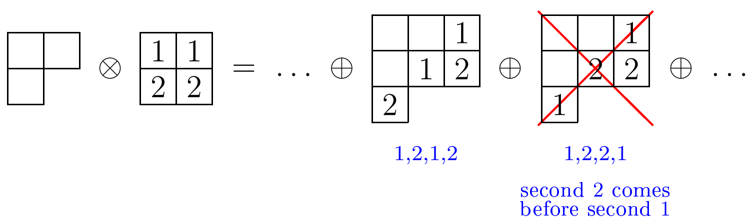
(c) When reading the numbers row-wise from *right to left* beginning with the first row, then the second etc., there must never be more is than $(i-1)$s in this sequence.

3. If two Young diagrams created in this way have the same shape, we only count them as different if the is are distributed differently.

4. For SU(N) columns with N boxes can be omitted.

5. Consistency check: compare dimensions on both sides of the equation!

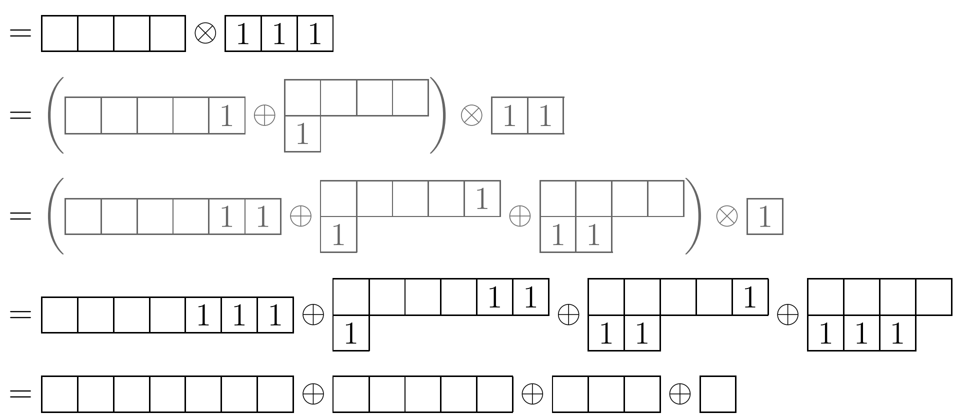
Illustration of rule 3c:



Examples:

1. SU(2)

$$5 \otimes 4 = (j{=}2) \otimes (j{=}\tfrac{3}{2})$$

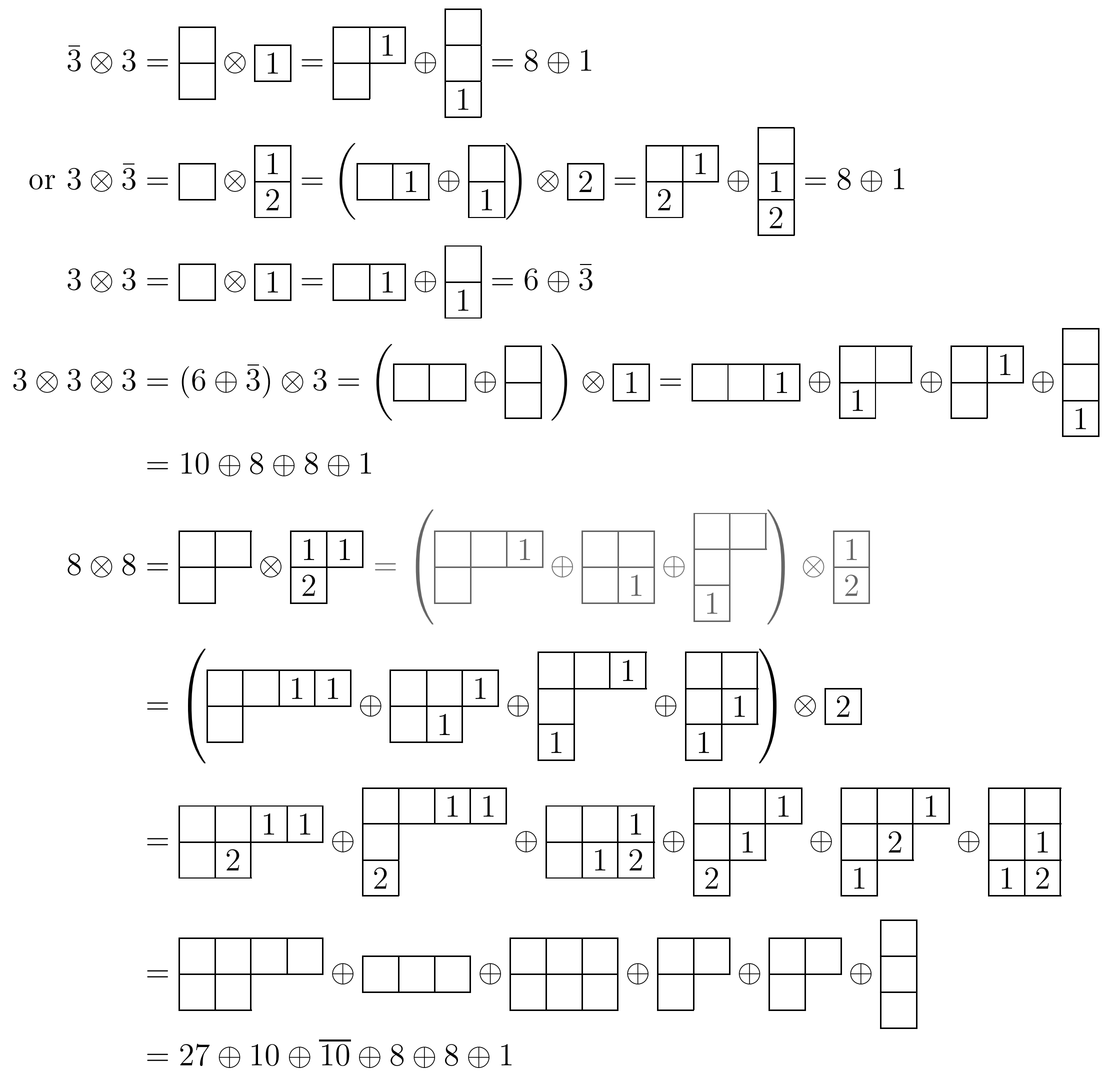


$$= 8 \oplus 6 \oplus 4 \oplus 2$$

$$= (j{=}\tfrac{7}{2}) \oplus (j{=}\tfrac{5}{2}) \oplus (j{=}\tfrac{3}{2}) \oplus (j{=}\tfrac{1}{2})$$

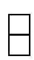
We obtained equivalent results in Problem 41 b) by different means.

2. SU(3)
Overbars in the following examples can be safely ignored; their meaning will be explained in the next section.



7.5 Complex conjugate representations

Observation: Sometimes $\dim\Gamma^{\lambda} = \dim\Gamma^{\lambda'}$ for $\lambda\neq\lambda'$. This may be "accidental" but often it can be understood systematically in terms of the following construction.

Example: Consider  for $N=3$.

Basis tensors: (anti-symmetric tensors of rank 2 in 3 dimensions)

$$|23\rangle - |32\rangle\,,\qquad |31\rangle - |13\rangle\,,\qquad |12\rangle - |21\rangle\,.$$

Action of GL(3), e.g.

$$
\begin{aligned}
g(|12\rangle - |21\rangle) &= |ij\rangle(g_{i1}g_{j2} - g_{i2}g_{j1}) \\
&= \underbrace{|23\rangle(g_{21}g_{32} - g_{22}g_{31}) + |32\rangle(g_{31}g_{22} - g_{32}g_{21})}_{=(|23\rangle-|32\rangle)\,\det\begin{pmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{pmatrix}} \\
&\quad + \underbrace{|31\rangle(g_{31}g_{12} - g_{32}g_{11}) + |13\rangle(g_{11}g_{32} - g_{12}g_{31})}_{=(|31\rangle-|13\rangle)\,(-1)\det\begin{pmatrix} g_{11} & g_{12} \\ g_{31} & g_{32}\end{pmatrix}} \\
&\quad + \underbrace{|12\rangle(g_{11}g_{22} - g_{12}g_{21}) + |21\rangle(g_{21}g_{12} - g_{22}g_{11})}_{=(|12\rangle-|21\rangle)\,\det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}},
\end{aligned}
$$

similarly for the other two basis elements. We find

$$
\Gamma^{\substack{\square\\\square}}(g) = \begin{pmatrix}
\det\begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{21} & g_{23} \\ g_{31} & g_{33}\end{pmatrix} & \det\begin{pmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{pmatrix} \\
(-1)\det\begin{pmatrix} g_{12} & g_{13} \\ g_{32} & g_{33}\end{pmatrix} & \det\begin{pmatrix} g_{11} & g_{13} \\ g_{31} & g_{33}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{11} & g_{12} \\ g_{31} & g_{32}\end{pmatrix} \\
\det\begin{pmatrix} g_{12} & g_{13} \\ g_{21} & g_{23}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{11} & g_{13} \\ g_{21} & g_{23}\end{pmatrix} & \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}
\end{pmatrix} = \mathrm{adj}(g)^T ,
$$

with the adjunct matrix $\mathrm{adj}(g)$. According to Cramer's rule $g^{-1} = \dfrac{\mathrm{adj}(g)}{\det g}$, i.e.

$$
\Gamma^{\substack{\square\\\square}}(g) = \det g \cdot (g^{-1})^T .
$$

Remark: This is true for arbitrary $N > 2$ and the Young diagram $\substack{\square\\\vdots\\\square}$ ($N-1$ boxes).

For SU(3) we have $\det g = 1$ and $g^{-1} = g^\dagger$, i.e. $\Gamma^{\substack{\square\\\square}}(g) = \overline{g}$. We write $\substack{\square\\\square} = \overline{\square}$ and also put a bar over the dimension

For GL(N), besides the defining rep g also $(g^{-1})^T$, $\overline{g}$ and $\overline{(g^{-1})^T}$ are N-dimensional irreps, in general non-equivalent.

For SU(N), due to $g^\dagger = g^{-1}$, we have

$$
(g^{-1})^T = \overline{g} \qquad \text{and} \qquad \overline{(g^{-1})^T} = g ,
$$

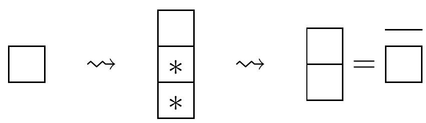
i.e. at most two of the four irreps are non-equivalent. For SU(2), even g and $\overline{g}$ are equivalent, see Problem 42; for $N \geq 3$ they are are non-equivalent. In terms of Young diagrams one obtains the complex conjugate irrep by means of the following procedure.

Complex conjugate representations for SU(N)

1. Consider a Young diagramm with at most $N-1$ rows. (The only m-row diagramm corresponds to the trivial rep which is identical to its complex conjugate.)
2. Add boxes to the Young diagram s.t. it becomes a rectangle of height N and same width as the original diagram.
3. Discard the original boxes and turn the added boxes by 180° – this is the Young diagram of the complex conjugate rep.

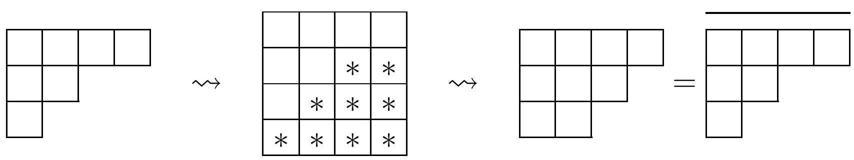
Examples:

1. SU(3)

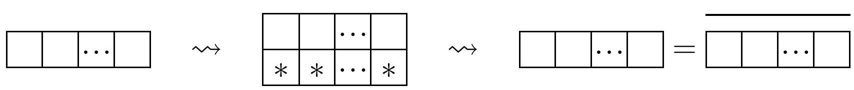


(see above)

2. SU(4)

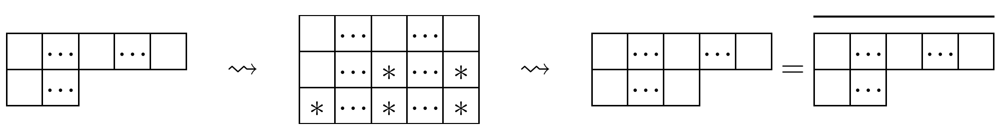


3. SU(2) in general



This is consistent with Problem 42, in which we showed, by other means, that for SU(2) every rep is equivalent to its complex conjugate.

4. SU(3) in general



i.e. $\overline{(\lambda_1, \lambda_2)} = (\lambda_1, \lambda_1 - \lambda_2)$.

8 Applications in particle physics

8.1 Elementary particles

- In the standard model of particle physics there are 3 (4) forces/interactions:
  1. strong (nuclear) force
  2. electromagnetic force
  3. weak (nuclear) force
  4. (gravitation)

  (2. & 3. together: electro-weak force)

- 3 (4) kinds of "elementary" particles:
  1. leptons (e.g. electron): spin $\frac{1}{2}$, do not interact via the strong force
  2. hadrons (e.g. proton, neutron): interact via the strong force
  3. particles which "carry" the forces (e.g. photon, gluon): integer spin
  4. Higgs boson

- Hadrons are composed of smaller particles (quarks with spin $\frac{1}{2}$) and come in two kinds:
  (a) baryons ($\sim qqq$, e.g. proton, neutron): spin $= \frac{1}{2}, \frac{3}{2}, \ldots$
  (b) mesons ($\sim \bar{q}q$, e.g. pions): spin $= 0, 1, 2, \ldots$

- lepton number:

$$L = \begin{cases} 1 & \text{for leptons} \\ -1 & \text{for anti-leptons} \\ 0 & \text{otherwise} \end{cases}$$

- baryon number:

$$B = \begin{cases} 1 & \text{for baryons} \\ -1 & \text{for anti-baryons} \\ 0 & \text{“otherwise”} \end{cases}$$

  quarks: $B = \frac{1}{3}$, anti-quarks: $B = -\frac{1}{3}$

8.2 SU(2) isospin

- experimental observation: Among hadrons we find sets ("multiplets"), with approximately the same mass (= eigenvalue of H),
  e.g. proton p and neutron n (baryons): $m_p \approx m_n \approx 940\,\text{MeV}$ or
  the three pions (mesons): $m_{\pi^0} \approx m_{\pi^+} \approx m_{\pi^-} \approx 140\,\text{MeV}$.

- theoretical explanation:
  - The strong force (essentially) determines the masses, and it is independent of the electrical charge.
  - The (small) mass differences (within a multiplet) come from the electro-weak force.

- The degenerate states should transform in an irrep of an "internal" symmetry group, which is initially unknown.
  $\rightsquigarrow$ Find a group which explains the observed particle (mass) spectrum,
  i.e. degrees of degeneracy = dimensions of irreps.

- Consider first p and n and define an object with two components, the nucleon,

$$N = \begin{pmatrix} p \\ n \end{pmatrix} .$$

  - Lives in a 2-dimensional space, called "isospin"-space.
  - Consider SU(2)-transformations on this space, with generators I_1, I_2, I_3.
  - p has $I_3 = \frac{1}{2}$, n has $I_3 = -\frac{1}{2}$ (by definition)
  - The Hamiltonian for the strong force commutes with all 3 generators, i.e.

$$[H, \vec{I}] = 0 .$$

    We say the strong force is invariant under $\mathrm{SU}(2)_{\text{isospin}}$.
  - N transforms in the 2-dimensional fundamental rep, or *doublet* rep ($I = \frac{1}{2}$) of $\mathrm{SU}(2)_{\text{isospin}}$.
  - Electrical charge Q is then given in terms of isospin by $Q = I_3 + \frac{1}{2}$.

- Other hadrons transform in different irreps of $\mathrm{SU}(2)_{\text{isospin}}$,

  e.g. the pions form an isospin *triplet* ($I = 1$) with
$$\begin{aligned} \pi^+ &: I_3 = 1 \\ \pi^0 &: I_3 = 0 \\ \pi^- &: I_3 = -1 . \end{aligned}$$

  - electrical charge doesn't fit to formula above $\rightsquigarrow$ postulate *hypercharge* Y (later U(1)) with

$$Q = I_3 + \tfrac{1}{2} Y .$$

    The nucleon (p and n) has $Y = 1$, the three pions have $Y = 0$.

- Different isospin multiplets are characterised by different values of quantum numbers related to the strong force (B, Y, I, $J =$ spin, $P =$ parity).
  For all particles within a multiplet these numbers are identical.

- H invariant under $\mathrm{SU}(2)_{\text{isospin}}$ does not only have consequences for masses, but also, e.g., for cross sections (via the Wigner-Eckart theorem and SU(2)-Clebsch-Gordan coefficients).

8.3 SU(2) flavour

…which, essentially, is still $\mathrm{SU}(2)_{\text{isospin}}$, but on the level of quarks.

- Hadrons are composed of quarks, whose interaction (strong force) is described by quantum chromodynamics (QCD).

- In nature we find 6 quark-"flavours" (u, d, s, c, b, t),
  of which 2 are 'very light' (u, d), one "light" (s), and 3 'heavy' (c, b, t).

- In experiments at low energies one observes only particles consisting of u and d.
  $\rightsquigarrow$ First consider only $N_f = 2$, i.e. a 2-dimensional flavour space.

- The reason for the isospin invariance of hadron masses is, that for $m_u = m_d$ the QCD Lagrangian is invariant under $\mathrm{SU}(2)_{\text{flavour}}$ transformations, i.e. the internal symmetry group is $\mathrm{SU}(2)_{\text{flavour}}$.

- The 2-dimensional fundamental rep of $\mathrm{SU}(2)_{\text{flavour}}$ acts on

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{array}{l} \text{up quark} \quad (I_3 = \frac{1}{2},\ Y = \frac{1}{3} \ \Rightarrow\ Q = \frac{2}{3})\,, \\ \text{down quark} \ (I_3 = -\frac{1}{2},\ Y = \frac{1}{3} \Rightarrow Q = -\frac{1}{3})\,, \end{array}$$

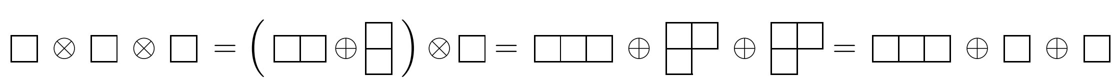
  i.e. q transforms as an doublet under $\mathrm{SU}(2)_{\text{flavour}}$ $(I = \frac{1}{2},\ Y = \frac{1}{3})$.
  (Thus, initially flavour is the same as isospin.)

- In the quark model the two nucleons have "quark content"

$$\begin{array}{ll} p \sim uud & (I_3 = \frac{1}{2}, Y = 1 \ \Rightarrow\ Q = 1) \\ n \sim udd & (I_3 = -\frac{1}{2}, Y = 1 \ \Rightarrow\ Q = 0) \end{array}$$

  ($\sim$ means we don't care about permutations of quarks at the moment,
  i.e. we now consider product states of the form $\square \otimes \square \otimes \square$.
  Here $\square$ denotes the 2-dimensional fundamental rep with $I = \frac{1}{2}$ and $Y = \frac{1}{3}$.

- Particles within a multiplet transform in an irrep $\rightsquigarrow$ decompose the product:



in terms of dimensions,

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$$

or in terms of the isospin quantum number I,

$$\tfrac{1}{2} \otimes \tfrac{1}{2} \otimes \tfrac{1}{2} = \tfrac{3}{2} \oplus \tfrac{1}{2} \oplus \tfrac{1}{2}\,.$$

In Section 8.4 we will see:

– The doublet $\begin{pmatrix} p \\ n \end{pmatrix}$ corresponds to a linear combination of the two doublets ($I = \frac{1}{2}$, $Y = 1$) on the r.h.s.

– The 4-dimensional irrep ($I = \frac{3}{2}$, $Y = 1$) corresponds to the Δ-baryons.

- Mesons consist – according to the quark model – of one quark and one anti-quark. The latter we obtain by applying the so-called charge conjugation operator $C = UK$. Here U is a unitary operator, and K is the (anti-unitary) operator of complex conjugation: (We don't care about U here – it acts on degrees of freedom which here play no role.)

$$Ku =: \bar{u} \qquad\qquad Kd =: \bar{d}$$

Consider an SU(2) transformation of the quark doublet:

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = g \begin{pmatrix} u \\ d \end{pmatrix} \qquad \Leftrightarrow \qquad \begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = \overline{g} \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}\,,$$

i.e. the "anti-doublet" $\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$ transforms in $\overline{2}$

Since for SU(2) $\bar{2}$ is equivalent to 2, we can also combine $\bar{u}$ and $\bar{d}$ into a doublet in such a way that it transforms in 2: With $h = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SU}(2)$ we have (cf. Problem 45)

$$\begin{aligned} \overline{g} &= h^{-1} g h \\ \begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} &= h^{-1} g h \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \\ h \begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} &= g\, h \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} \end{aligned}$$

and thus $h \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$ transforms in the same way as $\begin{pmatrix} u \\ d \end{pmatrix}$,
i.e. as an isospin doublet with

$$\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} \qquad \begin{matrix} (I_3 = \frac{1}{2},\ \ Y = -\frac{1}{3} \ \Rightarrow\ Q = \frac{1}{3})\,, \\ (I_3 = -\frac{1}{2},\, Y = -\frac{1}{3} \Rightarrow Q = -\frac{2}{3})\,, \end{matrix}$$

(Here we assumed, that $Y \mapsto -Y$ under charge conjugation.)

Now decompose

$$\square \otimes \bar{\square} = \underset{\substack{\uparrow \\ \binom{u}{d}}}{\square} \otimes \underset{\substack{\uparrow \\ \binom{-\bar{d}}{\bar{u}}}}{\square} = \square\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$$

$$\begin{array}{lll} \text{or} & 2 \cdot 2 \;= 3 + 1 & \text{(dimensions)} \\ \text{or} & \tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0 & \text{(isospin).} \end{array}$$

Construct multiplets as at the end of Section 7.2.2. There we had: triplet $= \{|11\rangle, \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), |22\rangle\}$, and singlet $= \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle)$.

– The isospin-triplet ($I = 1$, $Y = 0$) describes the the pions:

$$\begin{aligned} I_3 = 1 : &\qquad \pi^+ = -u\bar{d} \\ I_3 = 0 : &\qquad \pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\ I_3 = -1 : &\qquad \pi^- = d\bar{u} \end{aligned}$$

These states are invariant under $u \leftrightarrow -\bar{d}$, $d \leftrightarrow \bar{u}$.

– The singlet is the anti-symmetric linear combination of states which are tensor products of states with $I_3 = \frac{1}{2}$ und $I_3 = -\frac{1}{2}$, i.e.

$$\frac{1}{\sqrt{2}}(u\bar{u} - d(-\bar{d})) = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\,.$$

In Section 8.4 we will see that this describes the ω meson.

8.4 SU(3) flavour and the quark model

- At higher energies one also observes the strange quark.
  $\rightsquigarrow$ Consider now $N_f = 3$, i.e. a 3-dimensional flavour space with internal symmetry group $SU(3)_{\text{flavour}}$.
- additional quantum number: strangeness S, with $Y = S + B$

	B	I	I_3	Y	S	Q
u	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{2}{3}$
d	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
s	$\frac{1}{3}$	0	0	$-\frac{2}{3}$	-1	$-\frac{1}{3}$

- QCD processes leave S (and thus Y) invariant.
- The QCD-Lagragian (or Hamiltonian) is only invariant under $\mathrm{SU}(3)_{\text{flavour}}$, if $m_u = m_d = m_s$. Due to $m_u \approx m_d < m_s$, this symmetry is not exact, but broken to $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$.
  ⇒ No perfect degeneracy, but "small" mass differences between hadrons within an SU(3) multiplet (cf. Problem 49: Gell-Mann-Okubo formula for the baryon decuplet).

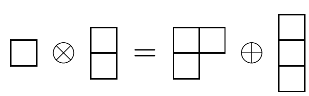
  Remark: The generators of SU(3) (a basis for the 8-dimensional Lie algebra $\mathfrak{su}(3)$ – traceless Hermitian 3×3 matrices) can be chosen s.t. (σ_j are the Pauli matrices)

$$\begin{pmatrix} \sigma_j & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}, \; j = 1,2,3, \qquad \text{and} \qquad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

  are among them. The first 3 generate $\mathrm{SU}(2)_I$ whereas the last one generates $\mathrm{U}(1)_Y$.
- The defining rep 3 of $\mathrm{SU}(3)_{\text{flavour}}$ acts on

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} .$$

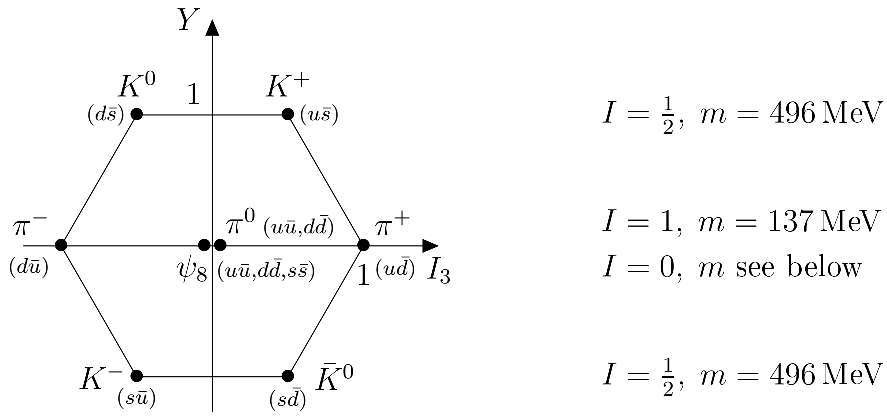
- Mesons consist of one quark and one anti-quark (which transforms in $\bar{3}$). Thus, decompose



$$\text{or} \qquad 3 \otimes \bar{3} \; = \; 8 \oplus 1 \, ,$$

  i.e. we expect multiplets of approximately (mass-)degenerate mesons consisting of 8 particles or one particle, respectively.
- Experimentally one finds: The lightest (i.e. ground state) mesons do actually form an octet and a singlet (together also called nonet), with quantum numbers $B = 0$ and $J^P = 0^-$. J is the usual spin.

– pseudoscalar meson-octet (scalar since $J = 0$, pseudo since $P = -1$):



(mass differences due to mass of strange quark)

– pseudoscalar meson-singlet: ψ_1 with $I = Y = 0$.

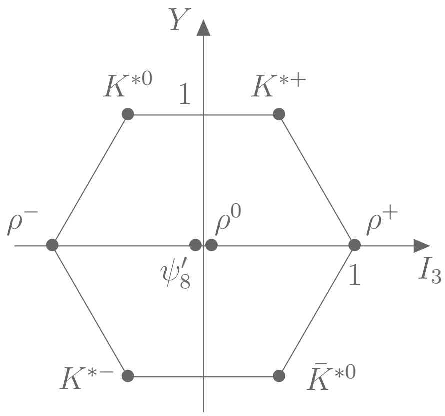
- It's slightly more complicated...
  - Consider all 3 states with $I_3 = Y = 0$:
    - π^0 is the $I_3 = 0$ state of the isospin-triplet, i.e. $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$.
    - ψ_1 is the SU(3)-singlet state, i.e. $\psi_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$.
    - ψ_8 is the SU(3)-octet, isospin-singlet state. orthogonal to both π^0 and ψ_1, $\psi_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$.
  - ψ_1 and ψ_8 have the same quantum numbers ($I = 0$ and $J^{PC} = 0^{-+}$).
    - If it was only for the strong interaction (QCD) then ψ_1 and ψ_8 would be physical states (transforming in different irreps of SU(3)).
    - Due to the electro-weak force these states can mix.

$$\eta(548\,MeV) = \psi_8 \cos\theta - \psi_1 \sin\theta$$
$$\eta'(958\,MeV) = \psi_8 \sin\theta + \psi_1 \cos\theta$$

The physical states (particles) are η and η'. θ is called nonet mixing angle (experimentally observed value (?) $\theta = -24.6°$).

- Furthermore, there are exited $q\bar{q}$-states (rotation, vibration etc.)
  The first "exited" meson-nonet has quantum numbers $B = 0$ and $J^P = 1^-$.

– vector meson-octet: (quark content as above)



$I=\frac{1}{2},\ m=892\,\mathrm{MeV}$

$I=1,\ m=776\,\mathrm{MeV}$

$I=0,\ m$ s.u.

$I=\frac{1}{2},\ m=892\,\mathrm{MeV}$

– vector meson-singlet: ψ'_1 with $I=Y=0$.

As above ψ'_1 and ψ'_8 mix, with $\theta_V = 36°$ (almost "ideal" mixing):

$$\phi(1020\,\mathrm{MeV}) = \psi'_8\cos\theta_V - \psi'_1\sin\theta_V \approx s\bar{s}$$
$$\omega(782\,\mathrm{MeV}) = \psi'_8\sin\theta_V + \psi'_1\cos\theta_V \approx \frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})$$

i.e. $$\underbrace{m_{\rho^0,\rho^+,\rho^-} \approx m_\omega}_{\text{no s-quark}} \quad < \quad \underbrace{m_{K^{*0},K^{*+},K^{*-},\bar{K}^{*0}}}_{\text{one s-quark}} \quad < \quad \underbrace{m_\phi}_{\text{two s-quarks}} .$$

- Baryons consist of 3 quarks. Thus, decompose

$$\square \otimes \square \otimes \square = \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}$$

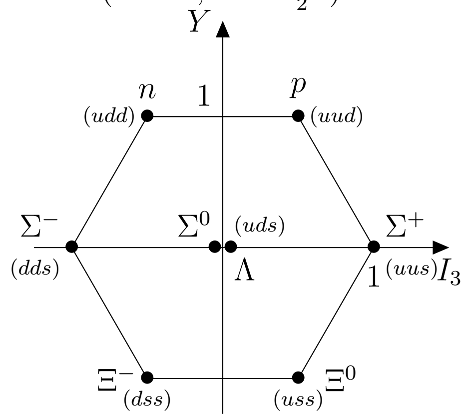
or $$3 \otimes 3 \otimes 3 = \underset{\substack{\uparrow \\ S}}{10} \oplus \underset{\substack{\uparrow \\ M_S}}{8} \oplus \underset{\substack{\uparrow \\ M_A}}{8} \oplus \underset{\substack{\uparrow \\ A}}{1} .$$

with S = tensors that are totally symmetric under S_3, i.e. under quark exchange,
M_S = tensors with mixed symmetry (symmetric under exchange of the first two quarks $*$),
M_A = tensors with mixed symmetry (anti-symmetric under exchange of the first two quarks, $*$),
A = totally anti-symmertic tensors.

$*$ This is different from what we get with Young operators for standard tableaux (symmetric under $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$, resp.), i.e. here we take linear combinations of those states.

We thus expect multiplets of (almost mass) degenerate baryons, consisting of 1, 8 or 10 particles.

- Experimentally one finds: The lightest (i.e. ground state) baryons form an octet and a decuplet:
  - baryon-octet ($B = 1$, $J^P = \frac{1}{2}^+$):



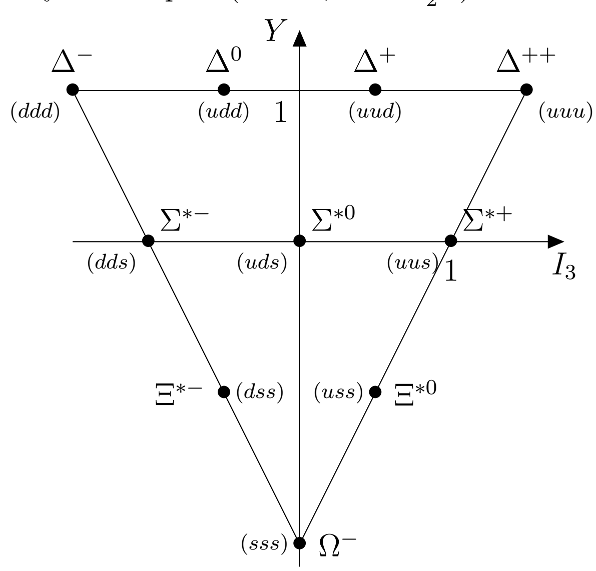
$I = \frac{1}{2},\ m = 939\,\text{MeV}$

$I = 1,\ m = 1193\,\text{MeV}$

$I = 0,\ m = 1116\,\text{MeV}$

$I = \frac{1}{2},\ m = 1318\,\text{MeV}$

  - baryon-decuplet ($B = 1$, $J^P = \frac{3}{2}^+$):



$I = \frac{3}{2},\ m = 1232\,\text{MeV}$

$I = 1,\ m = 1385\,\text{MeV}$

$I = \frac{1}{2},\ m = 1530\,\text{MeV}$

$I = 0,\ m = 1672\,\text{MeV}$

- What about the singlet and the second octet?
  - Baryons are fermions, and thus their total wave function (in space, spin, flavour and colour) have to be totally anti-symmetric.
  - Baryons are colour-singlets, i.e. they transform in the trivial rep $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ of $SU(3)_{\text{colour}}$, which is the sgn rep of S_3. $\Rightarrow$ The colour part of the wave function is totally anti-symmetric (under exchange of the quarks).
  - In the ground state orbital angular momentum is zero, i.e. the spatial part of the wave function is totally symmetric.

  $\Rightarrow$ The spin-flavour part has to be totally symmetric.

– For the spins of the 3 quarks in a baryon we have (Young diagrams for $SU(2)_{\text{spin}}$)

$$\square \otimes \square \otimes \square = \left(\square\!\square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) \otimes \square = \square\!\square\!\square \oplus \begin{smallmatrix}\square\!\square\\ \square\end{smallmatrix} \oplus \begin{smallmatrix}\square\!\square\\ \square\end{smallmatrix} = \square\!\square\!\square \oplus \square \oplus \square$$

or

$$2 \otimes 2 \otimes 2 = \underset{\substack{\uparrow\\ S}}{4} \oplus \underset{\substack{\uparrow\\ M_S}}{2} \oplus \underset{\substack{\uparrow\\ M_A}}{2} \qquad SU(2)_{\text{spin}}\,,$$

i.e. we have to combine

$$(\underset{\substack{\uparrow\\ S}}{10} \oplus \underset{\substack{\uparrow\\ M_S}}{8} \oplus \underset{\substack{\uparrow\\ M_A}}{8} \oplus \underset{\substack{\uparrow\\ A}}{1})_{\text{flavour}} \quad \text{and} \quad (\underset{\substack{\uparrow\\ S}}{4} \oplus \underset{\substack{\uparrow\\ M_S}}{2} \oplus \underset{\substack{\uparrow\\ M_A}}{2})_{\text{spin}}\,.$$

– This leads to the following possibilities for $(SU(3), SU(2))$-multiplets:

$$\begin{aligned} S &: (10,4),\ (8,2),\\ M_S &: (10,2),\ (8,4),\ (8,2),\ (1,2),\\ M_A &: (10,2),\ (8,4),\ (8,2),\ (1,2),\\ A &: (1,4),\ (8,2). \end{aligned}$$

Here the totally symmetric octet $(8,2)_S$ corresponds to the linear combination

$$(8,2)_S = \frac{1}{\sqrt{2}}[(\underset{\substack{\uparrow\\ M_S}}{8}\,,\ \underset{\substack{\uparrow\\ M_S}}{2}) + (\underset{\substack{\uparrow\\ M_A}}{8}\,,\ \underset{\substack{\uparrow\\ M_A}}{2})]\,,$$

and similarly for the other combinations.

– Only the totally symmetric spin-flavour multiplets $(10,4)$ and $(8,2)$ lead to totally symmetric wave function for the baryons.

$\Rightarrow$ In the ground state we have only one octet and the decuplet, but no singlet and no second octet. (In exited states, however, they can show up.)

- Alternative perspective:

– Each quark lives in 6-dimensional spin-flavour space (3 colours, 2 spin projections).

$\rightsquigarrow$ approximate $SU(6)$ spin-flavour symmetry.

– Decomposition into $SU(6)$-irreps:

$$6 \otimes 6 \otimes 6 = 56_S \oplus 70_{M_S} \oplus 70_{M_A} \oplus 20_A\,.$$

– The 56-dimensional irrep of $SU(6)$ induces a rep of $SU(3)_{\text{flavour}}$. The latter is reducible and we find

$$56_S = \underset{\substack{\nearrow\\ \dim = 10\cdot 4}}{10^{\frac{3}{2}}} \oplus \underset{\substack{\nwarrow\\ \dim = 8\cdot 2}}{8^{\frac{1}{2}}}\,.$$

This corresponds to the baryon-decuplet (spin $\frac{3}{2}$) and to the baryon-octet (spin $\frac{1}{2}$).

8.5 Gell-Mann-Okubo formula

- Within an $\mathrm{SU}(3)_{\text{flavour}}$-multiplet masses of particles within the same isospin-multiplet are almost identical, but for different Y (or S) mass differences can be larger.
  Reason: $m_u \approx m_d < m_s \;\Rightarrow\;$ $\mathrm{SU}(3)_{\text{flavour}}$ is broken to $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$.

- Assumption: The SU(3)-breaking term is a small perturbation,

$$H = H_0 + H' \,,$$

  with H_0 invariant under $\mathrm{SU}(3)_{\text{flavour}}$
  H' only invariant under $\mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$

- In Problem 49 we show using perturbation theory:

  - H' transforms like the ψ_8-state of the octet rep of SU(3) (cf. Section 8.4).
  - For the masses of baryons within a multiplet one finds the Gell-Mann-Okubo formula

$$m = a + bY + c\left(I(I+1) - \tfrac{1}{4}Y^2\right)$$

    with a, b, c constant within a multiplets. (In in Problem 49 we restrict our attention to rectangular Young diagram, in particular the decuplet; then there is no c.)

- This formula predicted the mass (of the then unknown) Ω^--particle, which was found a few years later with a mass within less than 1% of the prediction.

6 Lie groups (continued)

6.11 Roots and weights

Remark: Additive quantum numbers (examples: J_3 (spin), I_3 (isospin), Y hypercharge)
How did we draw the diagrams for the hadron multiplets in Section 8.4? We added that I_3- and Y-values for the quarks contributing to a hadron. This was justified because these values are eigenvalues of the two commuting generators...
Let G be a Lie group with Lie algebra $\mathfrak{g}$, let Γ^1 and Γ^2 be reps of G with corresponding reps $\mathrm{d}\Gamma^{1,2}$ of $\mathfrak{g}$. Consider $\Gamma = \Gamma^1 \otimes \Gamma^2$. Then

$$\mathrm{d}\Gamma(X) = \mathrm{d}\Gamma^1(X) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma^2(X)$$

since

$$\begin{aligned}
\mathrm{d}\Gamma(X) &= \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(\mathrm{e}^{\mathrm{i}Xt})\Big|_{t=0} = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma^1(\mathrm{e}^{\mathrm{i}Xt}) \otimes \Gamma(\mathrm{e}^{\mathrm{i}Xt})^2\right)\Big|_{t=0} \\
&= \frac{1}{\mathrm{i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^1(\mathrm{e}^{\mathrm{i}Xt}) \otimes \Gamma^2(\mathrm{e}^{\mathrm{i}Xt})\right)\Big|_{t=0} + \frac{1}{\mathrm{i}}\left(\Gamma^1(\mathrm{e}^{\mathrm{i}Xt}) \otimes \frac{\mathrm{d}}{\mathrm{d}t}\Gamma^2(\mathrm{e}^{\mathrm{i}Xt})\right)\Big|_{t=0} \\
&= \mathrm{d}\Gamma^1(X) \otimes \Gamma^2(e) + \Gamma^1(e) \otimes \mathrm{d}\Gamma^2(X)\,.
\end{aligned}$$

If ψ and φ are eigenvectors of $\mathrm{d}\Gamma^1(X)$ and $\mathrm{d}\Gamma^2(X)$, respectively, say

$$\mathrm{d}\Gamma^1(X)\psi = \lambda\psi\,, \qquad \mathrm{d}\Gamma^2(X)\varphi = \mu\varphi\,,$$

then

$$\mathrm{d}\Gamma(X)\,\psi \otimes \varphi = (\lambda + \mu)\,\psi \otimes \varphi\,.$$

(Same for (Young-)symmetrised tensor products, i.e. for linear combinations of tensor products with permuted factors.)

Recall: Representation theory of SU(2),
cf. Section 6.8 – where we actually started with SO(3),
generators / basis for $\mathfrak{su}(2)$: ($s_j = \frac{1}{2}\sigma_j$ with the Pauli matrices σ_j)

$$s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad s_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\,, \qquad s_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

with $[s_j, s_k] = \sum_\ell \mathrm{i}\varepsilon_{jk\ell}s_\ell$. Define

$$s_+ = s_1 + \mathrm{i}s_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \qquad s_- = s_1 - \mathrm{i}s_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,,$$

and conclude that

$$[s_3, s_\pm] = \pm s_\pm\,, \qquad [s_+, s_-] = 2s_3\,.$$

Consider a rep, $\mathrm{d}\Gamma(s_\bullet) =: J_\bullet$, with

$$J_3|m\rangle = m|m\rangle\,.$$

Then

$$J_3 J_\pm|m\rangle = (J_\pm J_3 + [J_3, J_\pm])|m\rangle = (J_\pm m \pm J_\pm)|m\rangle = (m \pm 1)J_\pm|m\rangle$$

The numbers m are called weights, and with $J_\pm$ we can raise and lower the weights if $J_\pm|m\rangle \neq 0$. If Γ is an irrep, then it is finite-dimensional, and then there has to be a highest (and lowest) weight, s.t. when we apply J_+ (J_-) it vanishes. This essentially fixed the representation theory of SU(2).

Continue with SU(3),
generators / basis for $\mathfrak{su}(3)$: $X_j = \frac{1}{2}\lambda_j$ with the Gell-Mann matrices

$$\lambda_k = \begin{pmatrix} \multicolumn{2}{c}{\sigma_k} & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for } k = 1,2,3, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

X_1, X_2, X_3 generate an SU(2) subgroup – and so do $X_4, X_5, \frac{1}{2}(\sqrt{3}X_8 + X_3)$ as well as $X_6, X_7, \frac{1}{2}(\sqrt{3}X_8 - X_3)$. Consequently we define

$$I_\pm = X_1 \pm \mathrm{i}X_2\,, \qquad U_\pm = X_6 \pm \mathrm{i}X_7\,, \qquad V_\pm = X_4 \pm \mathrm{i}X_5\,,$$

$$I_3 = X_3 \qquad \text{and keep} \quad X_8\,.$$

In physics one rather defines $Y = \frac{2}{\sqrt{3}}X_8$ for historical reasons. Then

$$[I_3, I_\pm] = \pm I_\pm\,, \qquad [I_3, U_\pm] = \mp\tfrac{1}{2}U_\pm\,, \qquad [I_3, V_\pm] = \pm\tfrac{1}{2}V_\pm\,,$$

$$[X_8, I_\pm] = 0\,, \qquad [X_8, U_\pm] = \pm\tfrac{\sqrt{3}}{2}U_\pm\,, \qquad [X_8, V_\pm] = \pm\tfrac{\sqrt{3}}{2}V_\pm\,.$$

For a rep Γ choose basis vectors as simultaneous eigenvectors of $\mathrm{d}\Gamma(X_3)$ and $\mathrm{d}\Gamma(X_8)$, say

$$\mathrm{d}\Gamma(I_3)|i_3, x_8\rangle = i_3|i_3, x_8\rangle\,, \qquad \mathrm{d}\Gamma(X_8)|i_3, x_8\rangle = x_8|i_3, x_8\rangle\,.$$

By a slight abuse of notation we omit $\mathrm{d}\Gamma$ in the following, i.e.

$$I_3|i_3, x_8\rangle = i_3|i_3, x_8\rangle\,, \qquad X_8|i_3, x_8\rangle = x_8|i_3, x_8\rangle\,.$$

Now

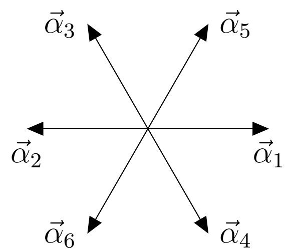
$$\begin{aligned}
I_3 I_\pm|i_3, x_8\rangle &= (i_3 \pm 1)I_\pm|i_3, x_8\rangle\,, & X_8 I_\pm|i_3, x_8\rangle &= x_8 I_\pm|i_3, x_8\rangle\,,\\
I_3 U_\pm|i_3, x_8\rangle &= (i_3 \mp \tfrac{1}{2})U_\pm|i_3, x_8\rangle\,, & X_8 U_\pm|i_3, x_8\rangle &= (x_8 \pm \tfrac{\sqrt{3}}{2})U_\pm|i_3, x_8\rangle\,,\\
I_3 V_\pm|i_3, x_8\rangle &= (i_3 \pm \tfrac{1}{2})V_\pm|i_3, x_8\rangle\,, & X_8 V_\pm|i_3, x_8\rangle &= (x_8 \pm \tfrac{\sqrt{3}}{2})V_\pm|i_3, x_8\rangle\,.
\end{aligned}$$

Now call the pairs $(i_3, x_8) =: \vec{m}$ weight vectors or simply weights (in our diagrams for hadron multiplets we indicated the positions of their tips as dots).

By applying reps of $I_\pm$, $U_\pm$ and $V_\pm$ we can shift the weights by

$$\vec{\alpha}_1 = (1,0)\,, \qquad \vec{\alpha}_2 = (-1,0)$$
$$\vec{\alpha}_3 = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\,, \qquad \vec{\alpha}_4 = (\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2})$$
$$\vec{\alpha}_5 = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\,, \qquad \vec{\alpha}_6 = (-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2})\,,$$

respectively. The vectors $\vec{\alpha}_j$ are called root vectors or simply roots. We collect them in a root diagram:



We call roots positive (negative), if their first component is positive (negative); same for weights. (If there was a root with vanishing first component, we would call it positive/negative according to the sign of the second component.) Hence $\vec{\alpha}_1, \vec{\alpha}_4$ and α_5 are positive.
Since irreps are finite-dimensional, there can be only finitely many weights for an irrep. Therefore, there has to be a highest (lowest) weight, which cannot be raised (lowered) by adding positive (negative) roots.

The adjoint rep: another route to roots. In the adjoint rep (rep of the Lie group on its own Lie algebra) we label can label also the basis vectors by generators,

$$\underbrace{\mathrm{ad}_{X_j}}_{\mathrm{d}\Gamma(X_j)} \underset{|\ldots\rangle}{\underset{\uparrow}{X_k}} = \underbrace{[X_j, X_k]}_{|\ldots\rangle}$$

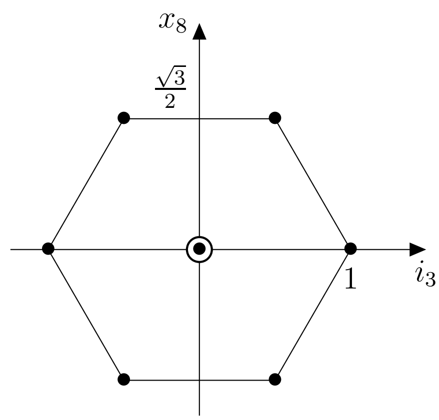
(Since we write generators as matrices, the bracket on the r.h.s. is a matrix commutator.)
Now

$$\mathrm{ad}_{I_3} I_3 = [I_3, I_3] = 0\,, \qquad \mathrm{ad}_{X_8} I_3 = [X_8, I_3] = 0\,,$$

i.e. the weight vector for the basis vector corresponding to I_3 is $(0,0)$; similarly the weight vector corresponding to X_8 is also $(0,0)$. Thus, the weight diagram for the adjoint rep of SU(3) has two points at the origin. Try to raise or lower weights from there, e.g.

$$\mathrm{ad}_{I_3}\mathrm{ad}_{I_\pm} I_3 = [I_3, [I_\pm, I_3]] = [I_3, \mp I_\pm] = \pm[I_\pm, I_3] = \pm\, \mathrm{ad}_{I_\pm} I_3\,,$$
$$\mathrm{ad}_{X_8}\mathrm{ad}_{I_\pm} I_3 = [X_8, [I_\pm, I_3]] \underset{\text{Jacobi id.}}{\underset{\uparrow}{=}} -[I_\pm, \underbrace{[I_3, X_8]}_{=0}] - [I_3, \underbrace{[X_8, I_\pm]}_{=0}]\,,$$

i.e. applying $I_\pm$ changes the weight by $(\pm 1, 0)$ – of course! We have to add the root vector $\vec{\alpha}_{1,2}$, as for any other rep (if the result is non-zero); similarly for $U_\pm$ and $V_\pm$. This already yields a weight diagram with eight (the dimension of $\mathfrak{su}(3)$) points, i.e. repeated attempts to raise or lower weights have to yield zero in the adjoint rep if the corresponding root vector would lead to a new point.

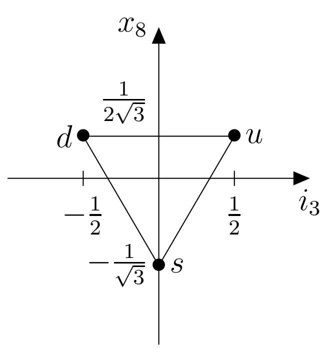


We can also verify explicitly that repeated application of the same raising or lowering operator to $(0,0)$ always yields zero,

$$\mathrm{ad}_{I_\pm}\mathrm{ad}_{I_\pm} I_3 = [I_\pm, [I_\pm, I_3]] = [I_\pm, \mp I_\pm] = 0\,,$$

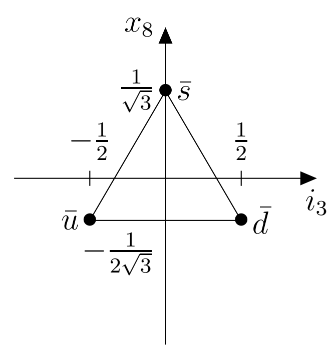
same if we replace I_3 by X_8 and/or $I_\pm$ by $U_\pm/V_\pm$.

The weight diagram of the defining rep is fixed by the diagonal elements of I_3 and X_8.



← defining rep

complex conjugate of defining rep →



For the complex conjugate of the defining rep we have to consider

$$\overline{\mathrm{e}^{\mathrm{i}X}} = \mathrm{e}^{-\mathrm{i}\overline{X}} = \mathrm{e}^{\mathrm{i}(-\overline{X})}\,,$$

i.e. $X \mapsto -X$; for our basis we have $X_{1,3,4,6,8} \mapsto -X_{1,3,4,6,8}$ and $X_{2,5,7} \mapsto X_{2,5,7}$, and in particular $(I_3, X_8) \mapsto (-I_3, -X_8)$, which fixes the weight diagram.

`Point out highest/lowest weights in all weight diagrams.`

6.12 From roots to the classification of semi-simple Lie algebras

Definition: (simple Lie group/algebra)
A Lie group G is called simple if it is connected, non-abelian, and has no nontrivial normal Lie subgroups. A Lie algebra $\mathfrak{g}$ is called simple if it is non-abelian and has no non-trivial ideals.

Remarks:

1. The Lie algebra of a simple Lie group is simple.
2. If $\mathfrak{g}$ is a simple Lie algebra then $\dim \mathfrak{g} \geq 2$.

Definition: (semi-simple Lie algebra)
A Lie algebra $\mathfrak{g}$ is called semi-simple if it is a direct sum of simple Lie algebras.

Remarks:

1. The Killing form of a semi-simple Lie algebra is non-degenerate.
2. Every Lie algebra is a semi-direct sum of something (its radical, i.e. its maximal solvable ideal – whatever that is) and a semi-simple Lie algebra.

The semi-simple Lie algebras can be classified completely in terms of their root systems.

In this final lecture I can only give a brief sketch of how this comes about.

Definition: A Cartan subalgebra of $\mathfrak{h}$ of a semi-simple Lie algebra $\mathfrak{g}$ is a maximal commutative subalgebra $\mathfrak{h}$ with ad_H diagonalisable $\forall\, H \in \mathfrak{h}$; $\dim \mathfrak{h}$ is called the rank of $\mathfrak{g}$. The rank is the maximal number of linearly independent, commuting, diagonalisable generators.

Weights

- Let G be a Lie group with Lie algebra $\mathfrak{g}$.
- Let $H_1, \ldots, H_\ell$ be a basis for $\mathfrak{h}$ (i.e. ℓ is the rank of G), hence

$$[H_j, H_k] = 0 \quad \forall\, j, k = 1, \ldots, \ell\,.$$

  The H_j are called Cartan generators; they are simultaneously diagonalisable.

- The eigenvalues m_j of H_j to a joint eigenvector are collected in a weight (vector) $\vec{m} = (m_1, \ldots, m_\ell)$.
- The weights for a fixed irrep are collected in a weight diagram (with possible degeneracies, cf. the $SU(3)$-octet). The number of weights in weight diagram is the dimension of the irrep. We can label basis vectors of irreducible subspaces by weights: $|\lambda, \vec{m}\rangle$.
- For SU(N) the generators are traceless (same for SO(N)).
  $\Rightarrow$ The sum of all weights in a weight diagram is $\vec{0}$ (for SU(N) or SO(N)).
- A weight is called positive (negative) if its first non-vanishing component is positive (negative).

- **Example:** SU(3), cf. Section 6.11
  - generators $X_1, \ldots, X_8$
  - commuting (Cartan) generators: X_3, X_8 (rank 2)
  - fundamental weights: (weight vectors of the defining rep)

$$\vec{m}_1 = (\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}})\,, \quad \vec{m}_2 = (-\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}})\,, \quad \vec{m}_3 = (0, -\tfrac{1}{\sqrt{3}})\,,$$

    notice: $\vec{m}_1 + \vec{m}_2 + \vec{m}_3 = \vec{0}$ (see Section 6.11 for the weight diagram)

- **Relation to Young diagrams**:
  SU(N) has rank $N-1$. For the irrep Γ^λ with Young diagram Θ_λ the weight diagram can be constructed as follows:
  - Label the boxes of Θ_λ by $j = 1, \ldots, n$ (i.e. let n be the number of boxes of Θ_λ).
  - Consider all ways in which we can write numbers $i_j = 1, \ldots, N$ into the boxes of Θ_λ, s.t. (cf. Section 7.2.5)
    * numbers within rows are non-decreasing, and
    * numbers within columns are increasing.
  - The weight vectors are then given by

$$\vec{M}^\lambda_{i_1 \cdots i_n} = \sum_{j=1}^{n} \vec{m}_{i_j}$$

    with the fundamental weights $\vec{m}_{i_j}$ (cf. the remark on additive quantum numbers at the beginning of Section 6.11).

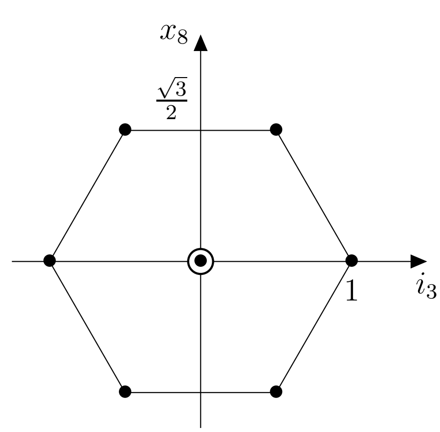
  Example: SU(3)-octet, i.e. $\Theta_\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array}$

  - 8 possibilities

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

  - corresponding weight vectors and weight diagram

$$\begin{aligned}
\vec{M}_{112} &= \vec{m}_1 + \vec{m}_1 + \vec{m}_2 = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \\
\vec{M}_{122} &= \vec{m}_1 + \vec{m}_2 + \vec{m}_2 = (-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) \\
\vec{M}_{132} &= \vec{m}_1 + \vec{m}_3 + \vec{m}_2 = (0, 0) \\
\vec{M}_{113} &= \vec{m}_1 + \vec{m}_1 + \vec{m}_3 = (1, 0) \\
\vec{M}_{123} &= \vec{m}_1 + \vec{m}_2 + \vec{m}_3 = (0, 0) \\
\vec{M}_{223} &= \vec{m}_2 + \vec{m}_2 + \vec{m}_3 = (-1, 0) \\
\vec{M}_{133} &= \vec{m}_1 + \vec{m}_3 + \vec{m}_3 = (\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}) \\
\vec{M}_{233} &= \vec{m}_2 + \vec{m}_3 + \vec{m}_3 = (-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2})
\end{aligned}$$



Roots

- Let $\mathfrak{g}$ be an n-dimensional semi-simple Lie algebra of rank ℓ.
- Recall (from Section 6.11) that in the adjoint rep we label both, reps of generators as well as basis vectors by generators,

$$\underbrace{\mathrm{ad}_{X_j}}_{\mathrm{d}\Gamma(X_j)} \underset{\underset{|\ldots\rangle}{\uparrow}}{X_k} = \underbrace{[X_j, X_k]}_{|\ldots\rangle},$$

for which we no introduce the shorthand notation

$$X_j|X_k\rangle = |[X_j, X_k]\rangle\,.$$

- Basis states corresponding to Cartan generators have weight zero,

$$H_j|H_k\rangle = |[H_j, H_k]\rangle = 0\,.$$

- The remaining $n - \ell$ basis states we call $|E_{\vec{\alpha}}\rangle$, labelled by their non-zero weights $\vec{\alpha}$ (non-zero since $[H_j, E_{\vec{\alpha}}] \neq 0$ for at least one j). The $|E_{\vec{\alpha}}\rangle$ can always be chosen as simultaneous eigenstates of the Cartan generators (without proof),

$$H_j|E_{\vec{\alpha}}\rangle = \alpha_j|E_{\vec{\alpha}}\rangle \qquad \Leftrightarrow \qquad [H_j, E_{\vec{\alpha}}] = \alpha_j E_{\vec{\alpha}}\,. \qquad (*)$$

  So far I concealed that we actually have to consider complex/complexified Lie algebras in this whole discussion, but recall (Section 6.11) that for SU(2) and SU(3) the raising and lowering operators were complex linear combinations of generators.

  Now $(*)$ implies

$$[H_j, E_{\vec{\alpha}}^\dagger] = -\alpha_j E_{\vec{\alpha}}^\dagger$$

  i.e. we can choose them s.t.

$$E_{\vec{\alpha}}^\dagger = E_{-\vec{\alpha}}\,. \qquad (+)$$

- The $n - \ell$ vectors $\vec{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$ are called root vectors or roots, i.e. the roots are the non-trivial weights of the adjoint rep.
  - Due to $(+)$ the number of roots is always even.
  - One can show that the roots are non-degenerate.
- The $E_{\vec{\alpha}}$ act as raising/lowering operators,

$$H_j E_{\vec{\alpha}}|E_{\vec{\beta}}\rangle = (E_{\vec{\alpha}} H_j + [H_j, E_{\vec{\alpha}}])|E_{\vec{\beta}}\rangle = (E_{\vec{\alpha}}\beta_j + \alpha_j E_{\vec{\alpha}})|E_{\vec{\beta}}\rangle = (\beta_j + \alpha_j)E_{\vec{\alpha}}|E_{\vec{\beta}}\rangle\,,$$

  i.e.

(i) $E_{\vec{\alpha}}|E_{\vec{\beta}}\rangle$ is proportional to $|E_{\vec{\alpha}+\vec{\beta}}\rangle$ if $\vec{\alpha}+\vec{\beta}$ is also a root,
$[E_{\vec{\alpha}}, E_{\vec{\beta}}]$ is proportional to $E_{\vec{\alpha}+\vec{\beta}}$ if $\vec{\alpha}+\vec{\beta}$ is also a root,

(ii) $[E_{\vec{\alpha}}, E_{-\vec{\alpha}}]$ is a linear combination of the H_j

(iii) $[E_{\vec{\alpha}}, E_{\vec{\beta}}] = 0$ if $\vec{\alpha}+\vec{\beta}$ is neither $\vec{0}$ nor a root.

In particular, if $\vec{\alpha}$ is a root then $2\vec{\alpha}$ cannot be a root (since $[E_{\vec{\alpha}}, E_{\vec{\alpha}}] = 0$).

- Now one considers the Jacobi identity for $E_{\vec{\alpha}}, E_{-\vec{\alpha}}, E_{k\vec{\alpha}+\vec{\beta}}$ and ... after calculating along for while ... one finds the condition

$$\frac{(\vec{\alpha},\vec{\beta})}{(\vec{\alpha},\vec{\alpha})} = \frac{\nu}{2} \quad \text{for some } \nu \in \mathbb{Z}\,.$$

Here the scalar product essentially shows up as

$$(\vec{\alpha},\vec{\beta}) = \sum_{j,k=1}^{\ell} \alpha_j \operatorname{tr}(H_j H_k)\beta_k\,,$$

and one can show that the generators can be chosen s.t.

$$\operatorname{tr}(H_j H_k) = \delta_{jk}\,, \qquad \operatorname{tr}(E_{\vec{\alpha}}E_{-\vec{\alpha}}) = \operatorname{tr}(E_{\vec{\alpha}}E_{\vec{\alpha}}^{\dagger}) = 1\,.$$

Interchanging the roles of $\vec{\alpha}$ and $\vec{\beta}$, one, of course, also finds

$$\frac{(\vec{\alpha},\vec{\beta})}{(\vec{\beta},\vec{\beta})} = \frac{\mu}{2} \quad \text{for some } \mu \in \mathbb{Z}\,.$$

Together the two conditions imply

$$\frac{(\vec{\alpha},\vec{\alpha})}{(\vec{\beta},\vec{\beta})} = \frac{\mu}{\nu} \qquad \text{and} \qquad \cos^2\theta = \frac{(\vec{\alpha},\vec{\beta})^2}{(\vec{\alpha},\vec{\alpha})(\vec{\beta},\vec{\beta})} = \frac{\nu\mu}{4}\,,$$

where θ is the angle between $\vec{\alpha}$ and $\vec{\beta}$. For $0 < \theta \leq 90°$ there are only four solutions to the second equation: 30°, 45°, 60°, 90° (i.e. $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$). The first condition fixes the corresponding length ratios, and together with some more symmetry conditions/restrictions this makes possible a complete classification of root systems and thus of semi-simple Lie algebras.