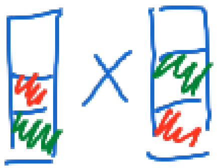
Kreuzprodukt

$$(\underline{a} \times \underline{b})_1 = \cdot - \cdot$$



Übungsklausur

1. a) Feldh. Gegenbsp: $\underline{f} = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$ C^1 auf $\mathbb{R}^3$

$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$, $t \in [0, 2\pi]$, hat $\int_\gamma \underline{f} \cdot d\underline{x} = \int_0^{2\pi} dt \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} = 2\pi$

1b) Falsch.

$$\frac{\partial f_1}{\partial y} = -(x+y)^{-2}$$

$$\frac{\partial f_2}{\partial x} = -2(x+y)^{-2} \neq \frac{\partial f_1}{\partial y}$$

1c) Richtig.

Das ist die Def.

1d) Falsch.

Weil $\mathbb{R}^3$ sternförmig,

Int. Bed. $\Rightarrow$ grad. $\Rightarrow$ kons.

1e) Falsch.

Das Möbiusband

ist nicht

or. bar.

A2 a)

$$a_i = \frac{1}{M} \int_F dS \, x_i \underbrace{\rho(\underline{x})}_{1}$$

$$M = \int_F \rho \, dS \overset{\text{hier}}{=} \int_F 1 \, dS = \mathrm{Fl}(F)$$

b) $r=1$, $\varphi \in [0, 2\pi]$, $\vartheta \in [0, \frac{\pi}{2}]$ nördl. Hemisph.

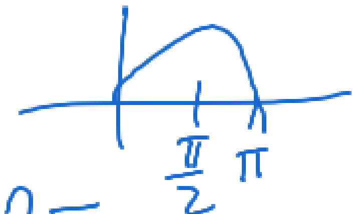
$$\phi(\varphi, \vartheta) = \begin{pmatrix} \cos\vartheta \cos\varphi \\ \cos\vartheta \sin\varphi \\ \sin\vartheta \end{pmatrix}, \quad \partial_\varphi \phi = \begin{pmatrix} -\cos\vartheta \sin\varphi \\ \cos\vartheta \cos\varphi \\ 0 \end{pmatrix},$$

$$\partial_\vartheta \phi = \begin{pmatrix} -\sin\vartheta \cos\varphi \\ -\sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}, \quad \partial_\varphi \phi \times \partial_\vartheta \phi = \begin{pmatrix} \cos^2\vartheta \cos\varphi - 0 \\ 0 + \cos^2\vartheta \sin\varphi \\ + \cos\vartheta \sin\vartheta \sin^2\varphi \cancel{\cos\varphi} + \cos\vartheta \sin\vartheta \cos^2\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\vartheta \cos\varphi \\ \cos^2\vartheta \sin\varphi \\ \cos\vartheta \sin\vartheta \end{pmatrix} = \cos\vartheta \, \phi(\varphi, \vartheta)$$

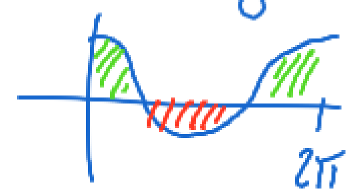
$$\|\partial_\varphi \phi \times \partial_\theta \phi\| = |\cos\theta| = \cos\theta$$

$$M = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \, \cos\theta = 2\pi \Big[\sin\theta \Big]_{\theta=0}^{\pi/2} = 2\pi$$



$a_1 = 0 = a_2$ aus Symmetrie

oder $$M a_1 = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \, \cos\theta \, \cos\theta \, \cos\varphi = \underbrace{\left(\int_0^{2\pi} d\varphi \, \cos\varphi\right)}_{0} \left(\int_0^{\pi/2} d\theta \, \cos^2\theta\right) = 0$$



$$\underbrace{M}_{2\pi} a_3 = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \cos\theta \sin\theta$$

$$(a+b)(a-b) = a^2 - b^2$$

$$= 2\pi \int_0^{\pi/2} d\theta \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \cdot \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)$$

$$= 2\pi \int_0^{\pi/2} d\theta \left(\frac{1}{4i} e^{2i\theta} - \frac{1}{4i} e^{-2i\theta} \right)$$

$$= 2\pi \left[\frac{1}{(2i)(4i)} e^{2i\theta} + \frac{1}{(+2i)(4i)} e^{-2i\theta} \right]_{\theta = 0}^{\theta = \pi/2}$$

$$= 2\pi \left(+\frac{1}{8} + (-1)\frac{1}{8}(-1) \right) = \frac{\pi}{\cancel{2}} \implies a_3 = \frac{1}{\cancel{4}\,2}$$

$$+\frac{1}{8} + \frac{1}{8}$$

$$\underline{a} = \begin{pmatrix} 0 \\ 0 \\ 1/\cancel{4}\,2 \end{pmatrix}.$$

3. $\phi(r,t) = \begin{pmatrix} r\cos t \\ r\sin t \\ t \end{pmatrix}$

$r \in [0,1]$
$t \in [0,2\pi]$

$$Fl = \int_F 1 \, dS = \int_0^1 dr \int_0^{2\pi} dt \; \| \partial_r \phi \times \partial_t \phi \|$$

$$\partial_r \phi = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \quad \partial_t \phi = \begin{pmatrix} -r\sin t \\ r\cos t \\ 1 \end{pmatrix}$$

$$\partial_r \phi \times \partial_t \phi = \begin{pmatrix} \sin t \\ -\cos t \\ r \end{pmatrix}, \quad \| \partial_r \phi \times \partial_t \phi \| = \sqrt{\sin^2 t + \cos^2 t + r^2}$$
$$= \sqrt{1+r^2}$$

$$\Rightarrow Fl = \int_0^1 dr \int_0^{2\pi} dt \sqrt{1+r^2} = 2\pi \int_0^1 dr \sqrt{1+r^2} \overset{\text{Tipp}}{=} \pi \left(\sinh^{-1}(1) + \sqrt{2} \right)$$

4. $\underline{f}(\underline{x}) := \underline{x}, \quad \mathbb{R}^3 \to \mathbb{R}^3, C^1$

$$\operatorname{div} \underline{f} = \sum_{i=1}^{3} \frac{\partial x_i}{\partial x_i} = 1+1+1 = 3$$

Gauß: $$\underbrace{\int_N \operatorname{div} \underline{f}\, d^3\underline{x}}_{\int_N 3\, d^3\underline{x} = 3\,\mathrm{Vol}(N)} = \int_{\partial N} \underline{x} \cdot d\underline{S}$$

$\Rightarrow$ Bh.

Wdh

- Wegintegral $\int_\gamma \underline{f}\cdot d\underline{x} = \int_a^b dt\; \underline{f}(\gamma(t))\cdot\gamma'(t)$ "$d\underline{x} = \gamma'(t)\,dt$"

$$\int_\gamma f\,ds = \int_a^b dt\; f(\gamma(t))\,\|\gamma'(t)\|$$ "$ds = \|\gamma'(t)\|\,dt$"

- $\nabla F = 0$ auf $G \implies F = \text{const.}$ auf jeder Zush. komp. von G.
- Def $\underline{f}$ kons. $\iff \int_\gamma \underline{f}\cdot d\underline{x}$ wegunabh.

Satz: G Gebiet: $\underline{f}$ kons. $\iff \underline{f}$ Gradient

Satz: G Gebiet: $\underline{f}$ Gradient $\implies$ Int. bed. $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$

Satz: G sternf. Gebiet: Int. bed. $\implies \underline{f}$ Gradient.

Schief geht auf $G =$  : Aharonov-Bohm $\underline{A}$ lokal Gradient

- Satz von Gauß ($B \subseteq \mathbb{R}^d$)

$$\int_B \operatorname{div} \underline{f}\, d^d\underline{x} = \int_{\partial B} \underline{f} \cdot d\underline{S}$$

(in 2d $\Longleftrightarrow$ Green)

- Satz von Stokes

3d $$\int_F \operatorname{rot} \underline{f} \cdot d\underline{S} = \int_{\partial F} \underline{f} \cdot d\underline{x}$$

(F or.)

2d $$\int_B \underbrace{(\operatorname{rot} \underline{f})_3}_{(\nabla \times \underline{f})_3 = \partial_1 f_2 - \partial_2 f_1} d^2\underline{x} = \int_{\partial B} \underline{f} \cdot d\underline{x}$$

(Satz von Green)

Rand-Orientierung

"B liegt linker Hand des Wanderers"

- "$d\underline{S} = \partial_u \phi \times \partial_v \phi \; du\, dv$"

  "$dS = \| \partial_u \phi \times \partial_v \phi \| \; du\, dv$"

  "$d\underline{S} = \underline{n} \; dS$", $\quad \underline{n} = \dfrac{\partial_u \phi \times \partial_v \phi}{\| \partial_u \phi \times \partial_v \phi \|}$.

- Transformationssatz für Integrale:

$$\int_U f(\underline{y})\, d^d\underline{y} = \int_{\varphi^{-1}(U)} f(\varphi(\underline{x}))\, J(\underline{x})\; d^d\underline{x} \qquad (\text{"}\underline{y} = \varphi(\underline{x})\text{"})$$

"$d^d\underline{y} = J(\underline{x})\, d^d\underline{x}$".

φ Diffeo, $J(\underline{x}) = |\det D\varphi(\underline{x})|$ Jacobi-Det.