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Sheet 1

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Matrix analysis. Part I

Exercise 1: Problem 1 of IMC2021

Let A be a real $n \times n$ matrix such that $A^3 = 0$.

1. Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A\,.$$

2. Express X in terms of A.

Exercise 2: Problem 5 of IMC2021

Let A be a real $n \times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that

$$2021B = A^m + B^2\,.$$

Prove that $|\det A| \leq 1$.

Exercise 3: Problem 2 of IMC2020

Let A and B be $n \times n$ real matrices such that

$$\mathrm{rk}(AB - BA + I) = 1\,,$$

where I is the $n\times$ n identity matrix.

Prove that

$$\mathrm{tr}(ABAB) - \mathrm{tr}(A^2B^2) = \frac{1}{2}n(n-1)\,.$$

($\mathrm{rk}(M)$ denotes the rank of matrix M, i.e., the maximum number of linearly independent columns in M; $\mathrm{tr}(M)$ denotes the trace of M, that is the sum of the diagonal elements in M.)