Universität Tübingen
Ángela Capel Cuevas

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Functions. Part I

Exercise 1: Problem 1 of IMC2016

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Suppose that f has infinitely many zeros, but there is no $x \in (a, b)$ with $f(x) = f'(x) = 0$.

1. Prove that $f(a)f(b) = 0$.
2. Give an example of such a function on $[0, 1]$.

Exercise 2: Problem 5 of IMC2020

Find all twice differentiable functions $f : \mathbb{R} \to (0, +\infty)$ satisfying

$$f''(x)f(x) \geq 2(f'(x))^2 .$$

for all $x \in \mathbb{R}$.

Exercise 3: Problem 6 of IMC2019

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $\big(f(0) - g'(0)\big)\big(g'(1) - f(1)\big) > 0$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = g'(c)$.

Exercise 4: Problem 7 of IMC2016

Consider a continuous function $f : [0, 1] \to \mathbb{R}$ satisfying $f(x) + f(y) \geq |x - y|$ for all pairs $x, y \in [0, 1]$. Find the minimum of $\int_0^1 f$ over all such functions.

Exercise 5: Problem 2 of IMC2017

Let $f : \mathbb{R} \to (0, \infty)$ be a differentiable function, and suppose that there exists a constant $L > 0$ such that

$$\left|f'(x) - f'(y)\right| \leq L\,|x - y|$$

for all x, y. Prove that

$$\big(f'(x)\big)^2 < 2Lf(x)$$

holds for all x.