IMC Training
SoSe 2022

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Number theory. Part I

Exercise 1: Problem 1 of IMC2020

Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

1. w consists of n letters, all of them from the alphabet $\{a, b, c, d\}$.
2. w contains an even number of letters a.
3. w contains an even number of letters b.

Exercise 2: Problem 1 of IMC2019

Evaluate the product:

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

Exercise 3: Problem 2 of IMC2015

For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(n) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \leq \frac{n^2}{4}.$$

When does equality hold?

Exercise 4: Problem 3 of IMC2016

Let n be a positive integer. Also, let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \ldots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} \leq \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i - \left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n} (a_i + b_i)}.$$

Exercise 5: Problem 6 of IMC2020

Find all prime numbers p for which there exists a unique $a \in \{1, 2, \ldots, p\}$ such that $a^3 - 3a + 1$ is divisible by p.