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Sheet 4

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Groups. Part I

Exercise 1: Problem 2 of IMC2018

Does there exist a field such that its multiplicative group is isomorphic to its additive group?

Exercise 2: Problem 6 of IMC2021

For a prime number p, let $GL_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\varphi : GL_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

Exercise 3: Problem 7 of IMC2020

Let G be a group and $n \geq 2$ be an integer. Let H_1 and H_2 be two subgroups of G that satisfy:

$$[G : H_1] = [G : H_2] = n \text{ and } [G : (H_1 \cap H_2)] = n(n-1)\,.$$

Prove that H_1 and H_2 are conjugate in G.

Note: Here, $[G : H]$ denotes the index of the subgroup H, i.e. the number of distinct left cosets xH of H in G. The subgroups H_1 and H_2 are conjugate if there exists an element $g \in G$ such that $g^{-1}H_1 g = H_2$.

Exercise 4: Problem 10 of IMC2012

Let $c \geq 1$ be a real number. Let G be an abelian group and let $A \subset G$ be a finite set satisfying $|A + A| \leq c|A|$, where $X + Y := \{x + y \,|\, x \in X,\, y \in Y\}$ and $|Z|$ denotes the cardinality of Z. Prove that

$$|\underbrace{A + A + \ldots + A}_{k \text{ times}}| \leq c^k |A|$$

for every positive integer k.