IMC Training
SoSe 2022

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Sheet 5

01. July 2022

Sequences and series. Part I

Exercise 1: Problem 1 of IMC2018

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

1. There is a sequence $\{c_n\}_{n=1}^{\infty}$ of positive numbers such that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge ;
2. $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges .

Exercise 2: Problem 6 of IMC2015

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2\,.$$

Exercise 3: Problem 7 of IMC2012

Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1$, $a_1 = 1/2$ and

$$a_{n+1} = \frac{n a_n^2}{1 + (n+1) a_n}\,, \qquad \text{for } n \geq 1.$$

Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

Exercise 4: Problem 6 of IMC2010

1. A sequence $x_1, x_2, \ldots$ of real numbers satisfies
$$x_{n+1} = x_n \cos x_n\,, \qquad \text{for all } n \geq 1\,.$$
Does it follow that this sequence converges for all initial values x_1?
2. A sequence $y_1, y_2, \ldots$ of real numbers satisfies
$$y_{n+1} = y_n \sin y_n\,, \qquad \text{for all } n \geq 1\,.$$
Does it follow that this sequence converges for all initial values y_1?

Exercise 5: Problem 3 of IMC2015

Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and $F(n) = \frac{5}{2}F(n-1) - F(n-2)$ for $n \geq 2$.

Determine whether or not $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F(2^n)}$ is a rational number.