Universität Tübingen
Ángela Capel Cuevas

Sheet 6

29. April 2022

Combinatory and optimization. Part I

Exercise 1: Problem 6 of IMC2018

Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \dots, v_k \in \mathbb{R}^n$ such that for every pair i, j of indices with $|i-j| > 1$ the vectors v_i and v_j are orthogonal.

Exercise 2: Problem 7 of IMC2011

An alien race has three genders: male, female and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual — if x likes y, then y likes x.

The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

1. Show that if n is even and $k = \frac{n}{2}$, then it might be impossible to create even one married triple.
2. Show that if $k \geq \frac{3n}{4}$, then it is always possible to create n disjoint married triples, thus marrying all of the expedition members.

Exercise 3: Problem 3 of IMC2012

Given an integer $n > 1$, let S_n be the group of permutations of the numbers $1, 2, \dots, n$. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group S_n. It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group S_n. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

Exercise 4: Problem 4 of IMC2016

Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties:

1. $\mathcal{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly k elements.
2. For any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$.

Prove that $\mathcal{F}$ contains at least three sets with at least n elements.

Exercise 5: Problem 8 of IMC2019

Let $x_1, \ldots, x_n$ be real numbers. For any set $I \subset \{1, 2, \ldots, n\}$ let $s(I) = \sum_{i \in I} x_i$. Assume that the function $I \mapsto s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1, 2, \ldots, n\}$. Prove that the number of sets $I \subset \{1, 2, \ldots, n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.