

Sheet 7

29. July 2022

Miscellanea. Part I

Exercise 1: Problem 1 of IMC2010

Let $0 < a < b$. Prove that

$$\int_a^b (x^2 + 1)e^{-x^2} dx \geq e^{-a^2} - e^{-b^2}. \quad (1)$$

Exercise 2: Problem 7 of IMC2010

Let a_0, a_1, \dots, a_n be positive real numbers such that $a_{k+1} - a_k \geq 1$ for all $k = 0, 1, \dots, n-1$. Prove that

$$1 + \frac{1}{a_0} \left(1 + \frac{1}{a_1 - a_0}\right) \dots \left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right) \left(1 + \frac{1}{a_1}\right) \dots \left(1 + \frac{1}{a_n}\right). \quad (2)$$

Exercise 3: Problem 1 of IMC2011

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. A point x is called a *shadow point* if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that

- All the points of the open interval $I = (a, b)$ are shadow points.
- a and b are not shadow points.

Prove that

1. $f(x) \leq f(b)$ for all $a < x < b$.
2. $f(a) = f(b)$.