IMC Training
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Test 1

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Exercise 1: Problem 1 of IMC1994

1. Let A be a $n \times n$, for $n \geq 2$, symmetric, invertible matrix with real positive elements. Show that
$$z_n \leq n^2 - 2n \,,$$
where z_n is the number of zero elements in A^{-1}.
2. How many zero elements are there in the inverse of the $n \times n$ matrix
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 1 & 2 & \dots & \dots \end{pmatrix} ?$$

a real $n \times n$ matrix such that $A^3 = 0$.

Exercise 2: Problem 2 of IMC1995

Let f be a continuous function on $[0,1]$ such that for every $x \in [0,1]$ we have
$$\int_x^1 f(t)dt \geq \frac{1-x^2}{2} \,.$$
Show that
$$\int_0^1 f^2(t)dt \geq \frac{1}{3} \,.$$

Exercise 3: Problem 5 of IMC1995

Let A and B be $n \times n$ real matrices. Assume that there exist $n+1$ different real numbers $t_1, t_2, \dots, t_{n+1}$ such that the matrices
$$C_i = A + t_i B \,, \quad i = 1, 2, \dots n+1 \,,$$
are nilpotent (i.e. $C_i^n = 0$).

Show that both A and B are nilpotent.

Exercise 4: Problem 2 of IMC1995

Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers such that $b_0 = 1$ and

$$b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}\,.$$

Calculate

$$\sum_{n=1}^{\infty} b_n 2^n\,.$$

Exercise 5: Problem 3, second day, of IMC1996

Let G be the subgroup of $GL_2(\mathbb{R})$, generated by A and B, where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let H consist of those matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in G for which $a_{11} = a_{22} = 1$.

1. Show that H is an abelian subgroup of G.
2. Show that H is not finitely generated.

Remark. *$GL_2(\mathbb{R})$ denotes the group of all 2×2 invertible matrices with real entries. Abelian means commutative. A group is finitely generated if there are a finite number of elements of the group such that every other element of the group can be obtained from these elements using the group operation.*

Exercise 6: Problem 4, second day, of IMC1996

Let B be a bounded closed convex symmetric (with respect to the origin) set in $\mathbb{R}^2$ with boundary the curve Γ. Let B have the property that the ellipse of maximal area contained in B is the disc D of radius 1 centered at the origin with boundary the circle C. Prove that $A \cap \Gamma \neq \emptyset$ for any arc A of C of length $l(A) \geq \frac{\pi}{2}$.