Quantum Shannon Theory and Beyond **Universität Tübingen**
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Data processing inequality for the relative entropy and the Petz recovery map

The aim of this project is to reproduce the proof of the data processing inequality for the relative entropy due to [2] and [7], we well as address the case in which equality is saturated in the inequality. That case was originally addressed in [4], [5], in which Petz showed that there is equality between the relative entropies of two density matrices before and after applying a quantum channel if, and only if, one density matrix can be recovered from the other by means of the (nowadays called) Petz recovery map.

In this project, we will follow the steps of [6] and [1] to prove the data processing inequality for the relative entropy under conditional expectations, a particular case of quantum channels. Next, we will study the case of equality in the data processing inequality and derive the Petz recovery map.

1 Preliminary notions

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on it. Let $\mathcal{S}(\mathcal{H})$ further denote the set of density matrices on $\mathcal{H}$. For simplicity in the calculations of the rest of the project, here we are going to restrict to the case of positive definite operators, instead of positive semi-definite operators. The other case is completely analogous, by taking pseudo-inverses instead of inverses and taking care of the corresponding inclusion of kernels of operators.

Definition. 1.1 (Relative entropy)**.** *The relative entropy of two positive states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ is given by*

$$D(\rho\|\sigma) := \operatorname{tr}[\rho(\log\rho - \log\sigma)]\,. \tag{1}$$

For the construction of the proof we want to present here, we will require the following equivalent formulation for the relative entropy. Let us denote by L_X, respectively R_X, the left-multiplication, resp. right-multiplication, operator by $X \in \mathcal{B}(\mathcal{H})$, i.e.

$$L_X(Y) = XY \quad , \qquad R_X(Y) = YX \tag{2}$$

for every $Y \in \mathcal{B}(\mathcal{H})$. Then, we can define the *modular operator* $\Delta_{\rho,\sigma} := L_\rho R_{\sigma^{-1}}$. Then, we can write the relative entropy of ρ and σ as

$$D(\rho\|\sigma) = \langle \rho^{1/2}\,, \log(\Delta_{\rho,\sigma})\rho^{1/2}\rangle\,. \tag{3}$$

Here, we are also going to restrict to a specific class of quantum channels, namely the conditional expectations.

Proposition. 1.2 ([3, Proposition 1.12]). *Let $\mathcal{M}$ be a matrix algebra with unital matrix subalgebra $\mathcal{N}$. Consider $\mathcal{H}$, $\mathcal{K}$ Hilbert spaces such that $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$. Then, there exists a unique linear mapping $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that*

1. *$\mathcal{E}$ is a positive map,*
2. *$\mathcal{E}(B) = B$ for all $B \in \mathcal{N}$,*
3. *$\mathcal{E}(AB) = \mathcal{E}(A)B$ for all $A \in \mathcal{M}$ and all $B \in \mathcal{N}$,*
4. *$\mathcal{E}$ is trace preserving.*

A map fulfilling (1)-(3) *is called a* conditional expectation*.*

It can be shown that conditional expectations are completely positive. Moreover, they are selfadjoint with respect to the Hilbert-Schmidt inner product. Note that proving the data processing inequality for these channels (and the corresponding conditions for saturation of equality) is enough to obtain the same results for every quantum channel, as Stinespring's dilation allows us to reconstruct any quantum channel from conditional expectations and isometries.

Theorem. 1.3 (Stinespring's dilation theorem). *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be two matrix algebras with Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, and let $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ be a quantum channel. Then, there exist a Hilbert space $\mathcal{V}$ and an isometry $V : \mathcal{H} \hookrightarrow \mathcal{K} \otimes \mathcal{V}$ such that*

$$\mathcal{T}(\omega) = \operatorname{tr}_{\mathcal{V}} [V \omega V^*]$$

for all states ω on $\mathcal{M}$. Here, $\operatorname{tr}_{\mathcal{V}}$ is the partial trace over the second system $\mathcal{V}$.

2 Data processing inequality

In this section, we want to prove the following: Given $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ a conditional expectation and two positive states $\rho, \sigma \in \mathcal{M}$, the following inequality holds

$$D(\rho \| \sigma) \geq D(\mathcal{E}(\rho) \| \mathcal{E}(\sigma)) \,. \tag{4}$$

We will divide the proof in several steps. First, let us define the map $U : \mathcal{M} \to \mathcal{M}$ given by

$$U(X) := \mathcal{E}(X) \sigma_{\mathcal{N}}^{-1/2} \sigma^{1/2} \,, \tag{5}$$

for all $X \mathcal{M}$, where we are denoting $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ (and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$). It is clear that its adjoint operator is given by

$$U^*(Y) := \mathcal{E}(Y \sigma^{1/2}) \sigma_{\mathcal{N}}^{-1/2} \,, \tag{6}$$

for all $Y \mathcal{M}$.

Exercise 1: Relation between $\Delta_{\rho,\sigma}$ and $\Delta_{\rho_{\mathcal{N}},\sigma_{\mathcal{N}}}$

Prove the following equality

$$U^* \Delta_{\rho,\sigma} U \leq \Delta_{\rho_{\mathcal{N}},\sigma_{\mathcal{N}}} \,. \tag{7}$$

The Schwarz inequality states that for any quantum channel $\mathcal{T}$, the following holds:

$$\mathcal{T}^*(X)\mathcal{T}^*(X^*) \leq \mathcal{T}^*(XX^*) \,. \tag{8}$$

Use it to prove $U^*U = \mathcal{E}$.

The matrix logarithm has the following integral representation

$$\log(A) = \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{t+A} \right) dt \tag{9}$$

Exercise 2: Data processing inequality

Use Eq. (3) along with Eq. (9) to write an integral representation for the relative entropy. Use such representation for the relative entropy to prove data processing inequality for $\mathcal{E}$. For that, use the relation between $\Delta_{\rho,\sigma}$ and $\Delta_{\rho_{\mathcal{N}},\sigma_{\mathcal{N}}}$ proven above, as well the fact that $x \mapsto (t+x)^{-1}$ is operator convex for every $0 \leq t$ jointly with Jensen inequality.

Hint: Note that a sufficient condition for Eq. (4) to hold is

$$\langle \rho^{1/2} \,, \, (t + \Delta_{\rho,\sigma})^{-1} \rho^{1/2} \rangle \geq \langle \rho_{\mathcal{N}}^{1/2} \,, \, (t + \Delta_{\rho_{\mathcal{N}},\sigma_{\mathcal{N}}})^{-1} \rho_{\mathcal{N}}^{1/2} \rangle \,. \tag{10}$$

3 Equality in data processing inequality

In this section, we want to prove that the following equality holds

$$D(\rho \| \sigma) = D(\mathcal{E}(\rho) \| \mathcal{E}(\sigma)) \,, \tag{11}$$

if, and only if,

$$\rho = \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2} \sigma^{1/2} \,. \tag{12}$$

Moreover, from the proof we will deduce that the roles of ρ and σ in the previous expression can be exchanged and both equalities are equivalent. In general, for any quantum channel $\mathcal{T}$, the map

$$\mathcal{R}_{\mathcal{T}}^{\sigma}(\cdot) := \sigma^{1/2} \mathcal{T}^* (\mathcal{T}(\sigma)^{-1/2} (\cdot) \mathcal{T}(\sigma)^{-1/2}) \sigma^{1/2} \,, \tag{13}$$

is called the *Petz recovery map* for the quantum channel $\mathcal{T}$ with respect to σ. Therefore, there is equality in the data processing inequality for $\mathcal{T}$ if, and only if,

$$\rho = \mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho) \,. \tag{14}$$

This is what we prove below.

For that, let us consider as a starting point the proof for the data processing inequality from the previous section. The difference of both relative entropies is composed by infinitely many slices indexed on time, which are all non-negative. Therefore, to have equality in Eq. (4), we need that each of this slices vanishes. This leads to the following:

Exercise 3: All slices vanish

Prove that Eq. (20) implies that the following identity is true:

$$U^*(\Delta_{\rho,\sigma}+t)^{-1}\sigma^{1/2} = (\Delta_{\rho_\mathcal{N},\sigma_\mathcal{N}}+t)^{-1}\sigma_\mathcal{N}^{1/2}\,, \tag{15}$$

for all $t>0$. Moreover, differentiate the previous expression and conclude that

$$\|U^*(\Delta_{\rho,\sigma}+t)^{-1}\sigma^{1/2}\|_2^2 = \|(\Delta_{\rho_\mathcal{N},\sigma_\mathcal{N}}+t)^{-1}\sigma_\mathcal{N}^{1/2}\|_2^2\,. \tag{16}$$

Since the last equality is of the form

$$\|U^*X\|_2 = \|X\|_2 \tag{17}$$

and U is a contraction, this implies that $UU^*X = X$ for every $X \in \mathcal{M}$. Use this to prove the following:

Exercise 4: Petz recovery map

Prove that the following holds:

$$U(\Delta_{\rho_\mathcal{N},\sigma_\mathcal{N}}+t)^{-1}\sigma_\mathcal{N}^{1/2} = (\Delta_{\rho,\sigma}+t)^{-1}\sigma^{1/2}\,. \tag{18}$$

Differentiate this expression n times and use Stone-Weierstrass approximation (for $x\mapsto x^{1/2}$) to show

$$\rho_\mathcal{N}^{1/2}\sigma_\mathcal{N}^{-1/2} = \rho^{1/2}\sigma^{-1/2}\,. \tag{19}$$

Conclude that

$$D(\rho\|\sigma) = D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \Rightarrow \rho = \sigma^{1/2}\sigma_\mathcal{N}^{-1/2}\rho_\mathcal{N}\sigma_\mathcal{N}^{-1/2}\sigma^{1/2}\,. \tag{20}$$

Exercise 5: Reverse direction

To completely conclude the result we want to show in this section, prove

$$D(\rho\|\sigma) = D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \Leftarrow \rho = \sigma^{1/2}\sigma_\mathcal{N}^{-1/2}\rho_\mathcal{N}\sigma_\mathcal{N}^{-1/2}\sigma^{1/2}\,. \tag{21}$$

References

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