Quantum Shannon Theory and Beyond **Universität Tübingen**
SoSe 2022 Ángela Capel Cuevas

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Part 1: Matrices and measurements

Exercise 1: Properties of matrices

Let $A, B \in \mathbb{C}^{n\times n}$ be two positive semi-definite matrices.

1. Show that for any $0 \leq t$, $A + tB$ is positive semi-definite.
2. Consider the matrix $M(t) := A + tB$ for $t > 0$. Show that there is a $t > 0$ such that $C(t) \geq 0$ if, and only if, $\operatorname{Ker}(A) \subseteq \operatorname{Ker}(B)$, where $\operatorname{Ker}(M) := \{x \in \mathbb{C}^n : Mx = 0\}$.
3. Let us recall that the exponential of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\exp(A) := \sum_{j=0}^{\infty} \frac{A^j}{j!} .$$

   Prove that, for any unitary matrix $U \in \mathbb{C}^{n\times n}$, the following holds:

$$\exp(UAU^*) = U \exp(A)\, U^* .$$

Exercise 2: Measurements

Let as consider a qubit, which undergoes three different scenarios:

1. We measure the qubit with X, and the outcome of the measurement is reported.
2. We measure the qubit with Z first, and the experiment continues only if the outcome is 1. If so, we measure the qubit with X, and the outcome of the measurement is reported.
3. We measure the qubit with Z first, and the outcome of the measurement is discarded. Next, we measure the qubit with X, and the outcome of the measurement is reported.

Consider the initial qubit to be

1. A pure state $|\varphi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}\,|1\rangle)$.
2. A mixed state $\rho = \frac{1}{3}(|0\rangle\langle 0| + 2\,|1\rangle\langle 1|)$.

Compute the probability of the measurement outcomes in the first and third scenarios for both possibilities of initial qubits. Next, compare the second and third scenarios to justify why discarding the measurement or observing it makes a difference in the subsequent measurement outcomes.

Exercise 3: Distinguishing between ensembles

Consider the following collections of states:

1. The states are either $|0\rangle$ or $|1\rangle$ with equal probability.
2. The states are either $\frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ or $\frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$ with equal probability.
3. The states are of the form $\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\phi}\,|1\rangle$, where θ is drawn uniformly at random from $[0,\pi]$ and ϕ is drawn uniformly at random from $[0,2\pi)$.

Is it possible to construct a measurement that can distinguish between these three ensembles?

Exercise 4: Thermal states

Consider the following two-qubit Hamiltonian:

$$H(\Delta) = X \otimes Z + \Delta(X \otimes I + I \otimes X)\,.$$

We want to study the properties of the thermal state ρ_β:

$$\rho(\Delta,\beta) = \frac{\mathrm{e}^{-\beta H(\Delta)}}{\mathrm{Tr}[\mathrm{e}^{-\beta H(\Delta)}]}$$

as a function of Δ and β.

1. Compute the eigenvalues of $H(\Delta)$.
2. Consider $\rho(\beta \to 0, \Delta)$. For which values of Δ is the thermal state separable, and for which entangled?
3. Consider $\rho(\beta \to \infty, \Delta)$. For which values of Δ is the thermal state separable, and for which entangled?

Part 2: Quantum circuits and teleportation

Exercise 5: Quantum gates

1. Show that
   a) $HXH = Z\,,$
   b) $HZH = X,$

   where
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2. Prove for every $\theta \in \mathbb{R}$ the following:

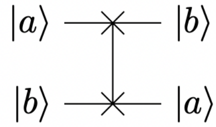
$$HR_x(\theta)H = R_z(\theta)\,,$$

where

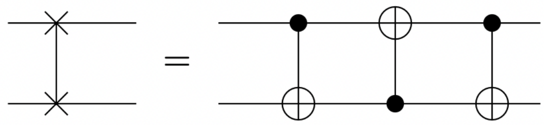
$$R_x(\theta) := \mathrm{e}^{-i\theta X/2}\,,$$
$$R_z(\theta) := \mathrm{e}^{-i\theta Z/2}\,.$$

Exercise 6: Basic quantum circuits

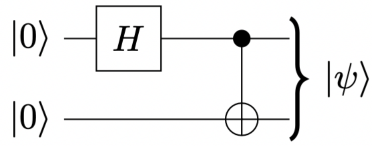
1. Express in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ the swap-gate, which maps $|a, b\rangle \mapsto |b, a\rangle$ and in circuit form is written as



2. Show that the swap-gate operation is equivalent to three CNOT gates, which is represented as



3. Compute the output $|\psi\rangle$ of the following circuit



4. Construct the CNOT gate from the controlled-Z gate and two Hadamard gates.

Exercise 7: Quantum teleportation protocol

Assume that Alice and Bob share an entangled state ρ_{AB}, ideally the Bell state:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\,.$$

Additionally, Alice has a single qubit state $|\phi_A\rangle$, which she would like to teleport to Bob. Let ρ_B the state received by Bob after the teleportation protocol. We can define the fidelity of this protocol as

$$F := \min_{|\phi_A\rangle} \langle\phi_A|\rho_B|\phi_A\rangle\,.$$

Compute this fidelity in the following two cases:

1. $\rho_{AB} = (1-\varepsilon) \left|\Phi_+\right\rangle \left\langle\Phi_+\right| + \varepsilon \frac{I}{4}$, where $\varepsilon \in (0,1)$.
2. $\rho_{AB} = \left|\Phi(\varepsilon)\right\rangle \left\langle\Phi(\varepsilon)\right|$, where $\left|\Phi(\varepsilon)\right\rangle = \sqrt{1-\varepsilon}\left|0,0\right\rangle + \sqrt{\varepsilon}\left|1,1\right\rangle$ and $\varepsilon \in (0,1)$.

Sketch these fidelities as functions of ε.