Quantum Shannon Theory and Beyond
SoSe 2022

Universität Tübingen
Ángela Capel Cuevas

Sheet 1 29. April 2022

Part 1: Brief review on Classical Information Theory

Exercise 1: Properties of entropies

1. Show that $0 \leq H(X) \leq \log_2(K)$.
2. Show that $H(X|Y) \geq 0$ and further
   a) $H(X,Y) \geq H(X)$ with equality if and only if $X = f(Y)$.
   b) $I(X:Y) \leq H(Y)$ with equality if and only if $X = f(Y)$.
3. **Subadditivity:** Show that
   $$H(X,Y) \leq H(X) + H(Y)\,,$$
   with equality if and only if X, Y are independent.
4. Show that
   $$H(X|Y) \leq H(X)\,,$$
   and thus $I(X:Y) \geq 0$ with equality if and only if X, Y are independent.
5. **Chain rule:** For $X_1, \ldots, X_n, Y$ random variables, show
   $$H(X_1, \ldots, X_n|Y) = \sum_{i=1}^{n} H(X_i|Y, X_1, \ldots, X_{i-1})\,.$$

Exercise 2: Fano's inequality

Let X and Y be two random variables and $\tilde{X} = f(Y)$ a function of Y with which we intend to guess the value of X. Let $p_e = p(X \neq \tilde{X})$ be the error made by guessing X with $\tilde{X}$. Then

$$H(p_e) + p_e \log(|X| - 1) \geq H(X|Y)\,.$$

Exercise 3: Flipping bits in strings

The number of strings flipping np bits is $nH(p)$, as the following approximation holds:

$$\log \binom{n}{np} \backsimeq nH(p)\,.$$

Part 2: Basic Notions on Quantum Information Theory

Exercise 4: Bra-kets and inner products

1. Write down the matrix representation for the following expressions:
   a) $|0\rangle\langle 1|$.
   b) $|0\rangle\langle 0| + |1\rangle\langle 1|$.
   c) $|+\rangle\langle 0|$.
2. Define the Hadamard gate as
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$
   Express it in the computational basis.
3. Compute $H|+\rangle$ using the bra-ket notation.
4. Let $|\varphi\rangle, |\psi\rangle \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$. Prove that the following holds:
$$\langle \varphi | A\psi \rangle = \langle A^* \varphi | \psi \rangle .$$
5. Prove that unitary matrices are norm-preserving.

Exercise 5: Tensor products

1. Consider two vectors $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}^2$. Express the product
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
   as a vector in $\mathbb{C}^4$.
2. Given two matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$, write an expression for
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$
   as a matrix of dimension 4×4.

Exercise 6: Bell states

Let us define the Bell states as
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) , \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) , \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) .$$
Show that they form an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Exercise 7: Pauli matrices

The Pauli matrices are given by:

$$I = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

1. Verify the following commutation relations:

$$[X, Y] = 2iZ \ , \quad [Y, Z] = 2iX \ , \quad [Y, Z] = 2iY \, .$$

2. Verify the following anticommutation relations:

$$A, B = 2\delta_{A,B} \ , \quad \text{for } A, B \in \{X, Y, Z\} \, .$$

3. Compute the eigenvalues of the Pauli matrices.

Exercise 8: Hermitian matrices of dimension 2×2

Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. Consider the following matrix:

$$A = \begin{pmatrix} \frac{1+\alpha}{2} & \beta \\ \beta^* & \frac{1-\alpha}{2} \end{pmatrix} .$$

1. Show that A is Hermitian and $\mathrm{tr}(A) = 1$.
2. Show that any Hermitian matrix in dimension 2×2 with unit trace can be expressed in this form.
3. Show that A can be uniquely expressed as a linear combination of the Pauli matrices.

Now, consider an arbitrary 2×2 Hermitian matrix ρ with unit trace, which we can write as

$$\rho = \frac{1}{2}(I + n_x X + n_y Y + n_z Z)$$

with $n_x, n_y, n_z \in \mathbb{R}$.

1. Compute the eigenvalues of ρ and express them in terms of n_x, n_y, n_z.
2. Denote $\mathbf{n} := (n_x, n_y, n_z)$. Show that $\rho \geq 0$ if, and only if, $|\mathbf{n}| \leq 1$.
3. Show that if $|\mathbf{n}| = 1$, then there are $v_1, v_2 \in \mathbb{C}$ such that

$$\rho = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} v_1^* & v_2^* \end{pmatrix} .$$