Quantum Shannon Theory and Beyond
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Sheet 4

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Approximating pure bipartite states by local operations

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a composite Hilbert space, where $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) =: d$ and consider a mixed density matrix ρ_{AB} on it. Assume that you can perform operations of the following form on the mixed state:

$$(\mathrm{id}_A \otimes T)(\rho_{AB}), \tag{1}$$

where $T : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_B)$ is a arbitrary quantum channel (that is, T is a completely positive and trace-preserving linear map). Your task now is to choose T such that the resulting density matrix is as similar as possible to $|\phi\rangle\langle\phi|$, for a certain pure state $|\phi\rangle \in \mathcal{H}_{AB}$. For our measure of similarity, we choose the Fidelity of $(\mathrm{id}_A \otimes T)(\rho_{AB})$ and $|\phi\rangle$, which is given by

$$\langle\phi| (\mathrm{id}_A \otimes T)(\rho_{AB}) |\phi\rangle . \tag{2}$$

Hence the optimal value of our problem is given by

$$\sup_{T \text{ channel}} \langle\psi|(\mathrm{id}_A \otimes T)(\rho_{AB})|\psi\rangle . \tag{3}$$

Exercise 1: Well defined?

Show that the supremum in (3) is attained by some channel T_{opt}.

In the following we will reformulate this problem as an SDP. To do so, we use the Choi-Jamiolkowski isomorphism.

Definition. 0.1. *The Choi matrix $C_T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of a linear map $T : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ is given by*

$$C_T := (T \otimes \mathrm{id}_2)(|\Omega\rangle\langle\Omega|), \tag{4}$$

where $|\Omega\rangle := \sum_{i=1}^{\dim(\mathcal{H}_2)} |ii\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2$, where each i labels an element of a fixed orthonormal basis of $\mathcal{H}_2$.

In the following, we are dealing with up to four copies of the same Hilbert space. We fix some basis $\{|1\rangle, |2\rangle, \ldots, |d\rangle\}$ in all of them. All basis dependent definitions (like $|\Omega\rangle$, C_T and the transpose) are defined w.r.t this basis.

Exercise 2: Properties of C_T

1. Show that for $\rho \in \mathcal{B}(\mathcal{H}_2)$,

$$T(\rho) = \mathrm{tr}_2[(\mathbb{1}_1 \otimes \rho^T)\, C_T], \tag{5}$$

   where tr_2 denotes the partial trace over the second system and ρ^T is the transpose of ρ.

2. Show that T is a quantum channel if and only if the following two relations hold:

$$C_T \geq 0, \quad \mathrm{tr}_1[C_T] = \mathbb{1}_2. \tag{6}$$

Conditions 2 already are constraints for an SDP. In order to formulate the whole problem as an SDP, we need to figure out how Choi matrices of channels of the form $\mathrm{id}_A \otimes T$ look like. The problem with writing this down directly now is that the tensor factors are not in the most convenient order. Therefore, we introduce the flip operator

$$\mathbb{F} := \sum_{i,j} |ij\rangle \langle ji| \tag{7}$$

Exercise 3: Choi matrix $C_{\mathrm{id}_A \otimes T}$

Show that

$$(\mathbb{1} \otimes \mathbb{F} \otimes \mathbb{1}) C_{\mathrm{id}_A \otimes T} (\mathbb{1} \otimes \mathbb{F} \otimes \mathbb{1}) = |\Omega\rangle \langle\Omega| \otimes C_T \tag{8}$$

Exercise 4: Objective function in terms of the Choi operator

Show that

$$\langle\phi| (\mathrm{id}_A \otimes T)(\rho_{AB}) |\phi\rangle = \mathrm{tr}[(\mathbb{1} \otimes \mathbb{F} \otimes \mathbb{1})(|\phi\rangle \langle\phi| \otimes \rho_{AB}^T)(\mathbb{1} \otimes \mathbb{F} \otimes \mathbb{1}) |\Omega\rangle \langle\Omega| \otimes C_T] \tag{9}$$

Exercise 5: Formulation of the SDP

Argue that the following optimization problem is an SDP, whose solution is equal to (3) and that the optimal C is the Choi operator of T_{opt}.

$$\begin{aligned} \text{maximize:} \quad & \mathrm{tr}[X(|\Omega\rangle \langle\Omega| \otimes C)] \\ \text{subject to:} \quad & C \geq 0 \\ & \mathrm{tr}_1[C] = \mathbb{1}_2, \end{aligned}$$

where $X := (\mathbb{1} \otimes \mathbb{F} \otimes \mathbb{1})(|\phi\rangle \langle\phi| \otimes \rho_{AB}^T)(\mathbb{1} \otimes \mathbb{F} \otimes \mathbb{1})$.