Quantum Shannon Theory and Beyond **Universität Tübingen**
SoSe 2022 Ángela Capel Cuevas

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Quantum channels

Exercise 1: Representation of quantum channels

Consider a single qubit channel which acts in the following way: With probability $p = 1/2$, it applies an X gate to the qubit (i.e. $X\rho X^*$), and with probability $p = 1/2$, it performs a measurement on the Z basis, after which:

- If the outcome of the measurement is $+1$, then it replaces the qubit with $|0\rangle$.
- If the outcome of the measurement is -1, then it replaces the qubit with $\mathbb{I}/2$.

Then, the following problems are posed:

1. Find a set of Kraus operators for this channel.
2. Find a Stinespring's dilation of this channel.
3. Find the Choi state corresponding to this channel.

Exercise 2: Quantum channel, Kraus representation and Lindbladian

Consider a quantum channel $\mathcal{E}$ on a qubit which measures the operator X on the qubit and:

- If the outcome of the measurement is $+1$, the qubit is erased and replaced with the state $\frac{1+\varepsilon_+}{2}|0\rangle\langle 0| + \frac{1-\varepsilon_+}{2}|1\rangle\langle 1|$.
- If the outcome of the measurement is $+1$, the qubit is erased and replaced with the state $\frac{1+\varepsilon_-}{2}|0\rangle\langle 0| + \frac{1-\varepsilon_-}{2}|1\rangle\langle 1|$.

In both cases, $\varepsilon_\pm \in (-1, 1)$.

1. Write an expression for $\mathcal{E}(\rho)$ in terms of ρ, where ρ is the single qubit state on which the map is applied. Show that the map is trace-preserving and positive.
2. Write down the Kraus representation for this channel.
3. Calculate the fixed points of this channel, i.e. the states ρ for which $\mathcal{E}(\rho) = \rho$.
4. Solve the master equation corresponding to the purely dissipative Lindbladian (with no Hamiltonian) with the jump operators $|0\rangle\langle +|$, $|1\rangle\langle -|$ and show that $\mathcal{E} = \lim_{t\to\infty} e^{\mathcal{L}t}$.

Exercise 3: Pure and product states

Let ρ be a density matrix for two qubits A and B such that the reduced state on qubit A is pure, i.e. $\mathrm{tr}_B(\rho) = |\psi\rangle\langle\psi|$ for some $|\psi\rangle$. Show that

1. If ρ is a pure state, then the reduced state on the qubit B is also pure.
2. ρ is a product state.

Exercise 4: PPT criterion

Consider the following state

$$\rho_p = p\,|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{I}{4}\,,$$

where $p \in (0,1)$, I is the identity operator and

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}\,.$$

1. Show that ρ_p is positive semidefinite for every $p \in (0,1)$.
2. Compute the partial transpose with respect to the second qubit of ρ_p.
3. Using the PPT criterion, calculate the range of p for which the state ρ_p is separable and when it is entangled.

Exercise 5: Transposition map

Consider the transposition map over $\mathbb{C}^{d\times d}$. Show that it is positive, but not completely positive.