Quantum Shannon Theory and Beyond **Universität Tübingen**
SoSe 2022 Ángela Capel Cuevas

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Quantum Hypothesis Testing

Exercise 1: Optimality under noise

Let $\rho_1, \ldots, \rho_n$ be density operators on $\mathbb{C}^d$ with associated a priori probabilities $p_x = 1/n$ for every $x = 1, \ldots, n$ and $M = (M_1, \ldots, M_n)$ a corresponding optimal measurement. Show that, for every $\lambda \in [0,1]$, a modified POVM M' is still optimal if we replace ρ_x by $\lambda \rho_x + (1-\lambda)\mathbf{1}/d$ with the same a priori proabilities.

Exercise 2: Distinguishing quantum states

Here we investigate the use of quantum hypothesis testing for distinguishing quantum states.

1. Given the two states $|0\rangle$ and $|+\rangle$, what is the optimal measurement for distinguishing them? Compute the corresponding probability.
2. Now change the second state by $p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$ with $p \in (0,1)$. How does the probability depend on p?
3. In 1., consider now several copies of the states, i.e. $|0000\rangle$ and $|++++\rangle$, for instance. How does the probability of discriminating the states successfully increase with the number of copies?

Exercise 3: Fidelity

The fidelity of two mixed states $\rho_A, \sigma_A \in \mathcal{S}(\mathcal{H}_A)$ can be defined by means of their purifications, namely

$$F(\rho_A, \sigma_A) := \max_{|\psi\rangle_{AB}, |\phi\rangle_{AB}} |\langle \psi | \phi \rangle_{AB}|^2 ,$$

where $|\psi\rangle_{AB}$ and $|\phi\rangle_{AB}$ are purifications of ρ_A and σ_A, respectively. Show that the following properties hold:

1. $F(\rho_A, \sigma_A) = 1$ if, and only if, $\rho_A = \sigma_A$.
2. $F(\rho_A, \sigma_A) = 0$ if, and only if, $\rho_A \sigma_A = \sigma_A \rho_A = 0$.
3. For any density operator $\tau \in \mathcal{S}(\mathcal{H}_C)$, we have $F(\rho_A \otimes \tau, \sigma_A \otimes \tau) = F(\rho_A, \sigma_A)$.

Exercise 4: Chernoff distance and fidelity

Let ρ, σ be two density matrices and denote $Q(s) := \mathrm{tr}[\rho^s \sigma^{1-s}]$ for $s \in [0,1]$, which defines the Chernoff distance by taking

$$- \inf_{s \in [0,1]} \log(Q(s)) \, .$$

1. Assuming that ρ and σ are pure states, what is the relation between $Q(s)$ and the fidelity $F(\rho, \sigma)$? Remember that, in general,

$$F(\rho, \sigma) := \left(\mathrm{tr} \left[(\rho^{1/2} \sigma \rho^{1/2})^{1/2} \right] \right)^2 \, .$$

2. Show that

$$\inf_{s \in [0,1]} Q(s) \leq F(\rho, \sigma)^{1/2} \, .$$

3. Show that for every $s \in [0,1]$,

$$F(\rho, \sigma) \leq Q(s) \, .$$

   Hint: It might help you to write

$$\rho^{1/2} \sigma^{1/2} = \rho^{\frac{1-s}{2}} \left(\rho^{\frac{s}{2}} \sigma^{\frac{1-s}{2}} \right) \sigma^{\frac{s}{2}} \, .$$

   and use that $\|AB\|_r \leq \|A\|_p \|B\|_q$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.