Quantum Shannon Theory and Beyond | **Universität Tübingen**
SoSe 2022 | Ángela Capel Cuevas

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Quantum entropies

Exercise 1: Duality of Conditional Entropy and bound

Let $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and consider a purification $|\psi\rangle_{ABE}$ of this state to some environment system E. Show that the coherent information satisfies:

$$I_\rho(A\rangle B) = H(B)_\psi - H(E)_\psi \,.$$

Additionally, show that

$$-H_\rho(A|B) = I_\rho(A\rangle B) = H_\psi(A|E)\,.$$

Moreover, prove the following upper bound for the conditional entropy:

$$|H_\rho(A|B)| \leq \log \dim(\mathcal{H}_A)\,.$$

Exercise 2: Bound on Quantum Mutual Information

Let $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Prove the following bound for the quantum mutual information:

$$I_\rho(A:B) \leq 2\,\log\left[\min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}\right]\,.$$

Give an example of a state that saturates the bound.

Exercise 3: Data Processing Inequality

Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ and $T : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ a quantum channel. The following inequality holds for the relative entropy:

$$D(\rho\|\sigma) \geq D(T(\rho)\|T(\sigma))\,.$$

Hint: Follow these steps in the proof:

1. *Use the fact that the relative entropy is jointly convex. Prove for $\rho_{AB}, \sigma_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ that*

$$D(\rho_{AB}\|\sigma_{AB}) \geq D(\rho_A\|\sigma_A)\,.$$

2. *Lift the previous inequality to any quantum channel using Stinespring's dilation. For that, use the fact that the relative entropy is consistent with the following limit:*

$$D(\rho\|\sigma) = \lim_{\varepsilon\to 0} D(\rho\|\sigma + \varepsilon\mathbf{1})\,,$$

for ρ and σ positive semi-definite.

Exercise 4: Data Processing Inequality implies other inequalities

Prove the strong subadditivity of the von Neumann entropy as a consequence of the previous exercise. For any $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$:

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})\,.$$

Exercise 5: Axiomatic Characterization of the Relative Entropy

Consider a function

$$f : \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \times \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \to [0, +\infty) \tag{1}$$

that satisfies the following properties:

- **Continuity:** $\rho \mapsto D(\rho\|\sigma)$ is continuous.
- **Additive:** For $\rho_A, \sigma_A \in \mathcal{S}(\mathcal{H}_A)$, $\rho_B, \sigma_B \in \mathcal{S}(\mathcal{H}_B)$, we have

$$f(\rho_A \otimes \rho_B\|\sigma_A \otimes \sigma_B) = f(\rho_A\|\sigma_A) + f(\rho_B\|\sigma_B)\,. \tag{2}$$

- **Superadditivity:** For $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$, $\sigma_A \in \mathcal{S}(\mathcal{H}_A)$, $\sigma_B \in \mathcal{S}(\mathcal{H}_B)$,

$$f(\rho_{AB}\|\sigma_A \otimes \sigma_B) \geq f(\rho_A\|\sigma_A) + f(\rho_B\|\sigma_B)\,. \tag{3}$$

- **Data processing inequality:** For $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, and T a quantum channel, we have

$$f(\rho\|\sigma) \geq f(T(\rho)\|T(\sigma))\,. \tag{4}$$

Then, f is the relative entropy.

Hint: Wilming, Gallego, Eisert - Axiomatic Characterization of the Quantum Relative Entropy and Free Energy - Entropy 2017, 19(6), 241