Sheet 8 30. June 2022

Quantum entropies. Part II

Exercise 1: Quantum Shearer's inequality

Consider the multipartite space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n$ and ρ a state in it. Prove the following inequality:

$$S(\rho) \geq \sum_i H(i|i^c)_\rho \,,$$

where

$$H(i|i^c)_\rho = S(\rho) - S(\rho_{1,2,\ldots,i-1,i+1,\ldots,n}) \,.$$

Hint: Use strong subadditivity.

Exercise 2: Triangle-like inequality

Consider $\rho, \sigma, \omega \in \mathcal{B}(\mathcal{H})$ and $\alpha \in [1/2, 1]$. The following inequality holds:

$$\widetilde{D}_\alpha(\rho\|\sigma) \leq \widetilde{D}_\alpha(\rho\|\omega) + D_{\max}(\omega\|\sigma) \,,$$

with

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \operatorname{tr} \left[\left(\rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{\alpha}} \rho^{\frac{1}{2}} \right)^\alpha \right]$$

Hint: Use Löwner-Heinz's theorem.

Exercise 3: Sandwiched Rényi divergences

Let $\rho, \sigma \in \mathcal{B}(\mathcal{H})$ and let $\mathcal{P}_{\sigma^{\otimes n}}$ be the Pinching map with respect to $\sigma^{\otimes n}$. Prove that the sandwiched Rényi divergence of ρ and σ with $\alpha \geq 0$ can be obtained in the following form:

$$\widetilde{D}_\alpha(\rho\|\sigma) = \lim_{n \to \infty} D_\alpha(\mathcal{P}_{\sigma^{\otimes n}}(\rho^{\otimes n})\|\sigma^{\otimes n}) \,,$$

where the term in the right-hand side can be viewed as a classical Rényi divergence.

Hint: Follow the next steps

1. *Prove that for $\alpha > 1$, the following inequality holds:*

$$\widetilde{Q}_\alpha(\rho\|\sigma) \geq \widetilde{Q}_\alpha(\mathcal{P}_\sigma(\rho)\|\sigma) \,.$$

   For $\alpha < 1$, we have the opposite direction.

2. *For $\rho \leq \rho'$, we have*

$$\widetilde{Q}_\alpha(\rho\|\sigma) \leq \widetilde{Q}_\alpha(\rho'\|\sigma)\,.$$

For $\sigma \leq \sigma'$, we have for $\alpha \geq 1$

$$\widetilde{Q}_\alpha(\rho\|\sigma) \geq \widetilde{Q}_\alpha(\rho\|\sigma')\,,$$

whereas for $1/2 \leq \alpha \leq 1$, we have the opposite direction.

3. *For $\alpha \in (1,2]$, we have*

$$\widetilde{Q}_\alpha(\rho\|\sigma) \leq |\mathrm{spec}(\sigma)|^{\alpha-1}\widetilde{Q}_\alpha(\mathcal{P}_\sigma(\rho)\|\sigma)\,.$$

The opposite direction holds for $\alpha \in (0,1]$.